# A Family of Pulse-Shaping Filters with ISI-Free Matched and Unmatched Filter Properties

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***Abstract*—The raised-cosine pulse-shaping filter plays an important role in digital communications due to its intersymbol interference (ISI)-free property. The ISI-free property holds after matched filtering is performed. In this letter, we propose a new family of pulse-shaping filters. These filters are ISI free with or without matched filtering. Using these new pulse-shaping filters, the computational load, and therefore the hardware cost in demodulation for modem design, might be reduced in some applications.**

***Index Terms*—ISI-free property, matched and unmatched filtering, pulse-shaping filters.**

## I. INTRODUCTION

THE raised-cosine filter

$$H(\omega)=\begin{cases}1, & 0\le w\le \frac{\pi}{T_s}(1-\alpha)\\ \cos^2\left[\frac{T_s}{4\alpha}\left(\omega-\frac{\pi(1-\alpha)}{T_s}\right)\right], & \frac{\pi}{T_s}(1-\alpha)\le\omega\le\frac{\pi}{T_s}(1+\alpha)\\ 0, & \omega>\frac{\pi}{T_s}(1+\alpha)\end{cases} \quad (1)$$

plays an important role in digital communication systems. It has been used extensively in modem design for both wireline and radio systems. This is mainly due to its intersymbol interference (ISI)-free property, i.e.,

$$h(nT_s)=\delta(n)=\begin{cases}1, & n=0\\ 0, & n=\pm1,\pm2,\cdots\end{cases}$$

where $H(\omega)$ and $h(t)$ are the frequency and the time response functions, respectively. There have been extensive discussions of this topic; see, for example, [1]–[4].

Since the ISI-free property holds after the matched filtering is performed for the received signal, the frequency response $G(\omega)$ of the transmitted waveform $g(t)$ should be the square root of $H(\omega)$ in (1), i.e.,

$$G(\omega)=\sqrt{H(\omega)} \quad \text{and} \quad g(t)=\mathcal{F}^{-1}(G(\omega)) \quad (2)$$

where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ means its inverse. The matched filtering plays two roles here. One is low-pass filtering that reduces the noise, and the other is ISI reduction due to the ISI-free property of the raised-cosine filters. Since the length of these filters is not short, the hardware implementation cost in current modem systems is significant. However, it may occur in practice that, for some users, the matched filtering is used purely for reducing the ISI. In this case, if the transmitted signal is already ISI free, the matched filtering may not be necessary. The question then becomes whether it is possible to construct pulse shaping filters $G(\omega)$ at the transmitter so that both the transmitted signal and the signal after matched filtering are ISI free, i.e.,

$$g(nT_s)=\delta(n) \quad \text{and} \quad h(nT_s)=\delta(n)$$

where $h(t)$ is the time-domain waveform of $H(\omega)=|G(\omega)|^2$.

In this letter, we will positively answer this question by proposing a family of such pulse-shaping filters.

## II. A NEW FAMILY OF PULSE-SHAPING FILTERS

In this section, we present a new family of real-valued pulse-shaping filters which have ISI-free properties with or without matched filtering.

Let $g(t)$ denote the waveform in the time domain to be transmitted, and let $G(\omega)$ denote its Fourier transform. Let $h(t)$ be the waveform in the time domain after the matched filtering of $g(t)$ is performed, and let $H(\omega)$ denote its Fourier transform. Then, $H(\omega)=|G(\omega)|^2$. Without loss of generality, from now on, we assume $T_s=1$. The ISI-free property for the waveform $g(t)$ is

$$g(n)=\delta(n), \qquad n\in\boldsymbol{Z}$$

where $\boldsymbol{Z}$ is the set of all integers. This is equivalent to

$$\sum_n G(\omega+2n\pi)=1. \quad (3)$$

The ISI-free property for the waveform $h(t)$ is

$$h(n)=\delta(n), \qquad n\in\boldsymbol{Z}$$

which is equivalent to

$$\sum_n |G(\omega+2n\pi)|^2=1. \quad (4)$$

We want to construct real-valued $g(t)$ that satisfies (3) and (4).

Let $\nu(x)$ be a continuous function such that

$$\nu(x)=\begin{cases}0, & x\le 0\\ 1, & x\ge 1\end{cases} \quad (5)$$

and

$$\nu(x)+\nu(1-x)=1, \qquad x\in\boldsymbol{R} \quad (6)$$

where $\boldsymbol{R}$ is the set of all real numbers. An example of such a $\nu(x)$ is

$$\nu(x)=\begin{cases}0, & x\le 0\\ x^4(35-84x+70x^2-20x^3), & 0\le x\le 1\\ 1, & x\ge 1\end{cases} \quad (7)$$

Paper approved by K. Townsend, the Editor for Computer-Aided Design of Communications Systems of the IEEE Communications Society. Manuscript received January 21, 1997; revised May 5, 1997. This work was supported in part by an initiative grant from the Department of Electrical Engineering, University of Delaware, the Air Force Office of Scientific Research (AFOSR) under Grant F49620-97-1-0253, and the National Science Foundation CAREER Program under Grant MIP-9703377.

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Publisher Item Identifier S 0090-6778(97)07267-X.

which has almost fourth-order smoothness. The simplest form for such $\nu$ is

$$\nu(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$

which is only continuous, but not differentiable.

We determine $g(t)$ by constructing its Fourier transform $G(\omega)$:

$$G(\omega) = \begin{cases} 1, & |\omega| \le \frac{2}{3}\pi \\ \frac{1}{2}\left(1 + e^{j\pi\nu((3/2\pi)\omega - 1}\right), & \frac{2}{3}\pi < \omega < \frac{4}{3}\pi, \\ \frac{1}{2}\left(1 - e^{j\pi\nu((3/2\pi)(\omega+2\pi)-1)}\right), & -\frac{4}{3}\pi < \omega < -\frac{2}{3}\pi \\ 0, & |\omega| \ge \frac{4}{3}\pi. \end{cases} \tag{8}$$

Notice that the parameter function $\nu$ controls the width of the transfer band of the filter $G(\omega)$. The smoothness of the function $\nu$ determines the speed of the waveform decay of $g(t)$ in the time domain, i.e., the length of the filter. The smoother $\nu$ is implies the shorter the filter $g(t)$ will be.

*Theorem 1:* The pulse-shaping filters $g(t)$ defined by (8) satisfy the following properties.

1) They are real valued.
2) They are ISI free by themselves, i.e., $g(n) = \delta(n)$.
3) They are ISI free after matched filtering is performed, i.e., $h(n) = \delta(n)$.

*Proof:* To prove 1), we only need to prove $G^*(-\omega) = G(\omega)$ for $2\pi/3 < |\omega| < 4\pi/3$

$$\begin{aligned} G^*(-\omega) &= \tfrac{1}{2}\left(1 - e^{-i\pi\nu((3/2\pi)(-\omega+2\pi)-1)}\right) \\ &= \tfrac{1}{2}\left(1 - e^{-i\pi\nu(2-(3/2\pi)\omega)}\right) \\ &\stackrel{1}{=} \tfrac{1}{2}\left(1 - e^{-i\pi(1-\nu(-1+(3/2\pi)\omega))}\right) \\ &= \tfrac{1}{2}\left(1 + e^{i\pi\nu(3/2\pi)\omega-1)}\right) \\ &= G(\omega) \end{aligned}$$

where step 1 is from (6).

To prove 2), we only need to prove (3). The form of $G(\omega)$ in (8) satisfies (3) for $2\pi/3 < \omega < 4\pi/3$

$$\sum_n G(\omega + 2n\pi) = G(\omega) + G(\omega - 2\pi) = 1.$$

This proves 2).

The property 3) can be similarly proved. □

The frequency responses $H(\omega)$ and $G(\omega)$ for the above new pulse-shaping filters in (8) with the $\nu$ function in (7), and the raised-cosine filter with $\alpha = 1/3$ in (1) and its square root are illustrated in Fig. 1.

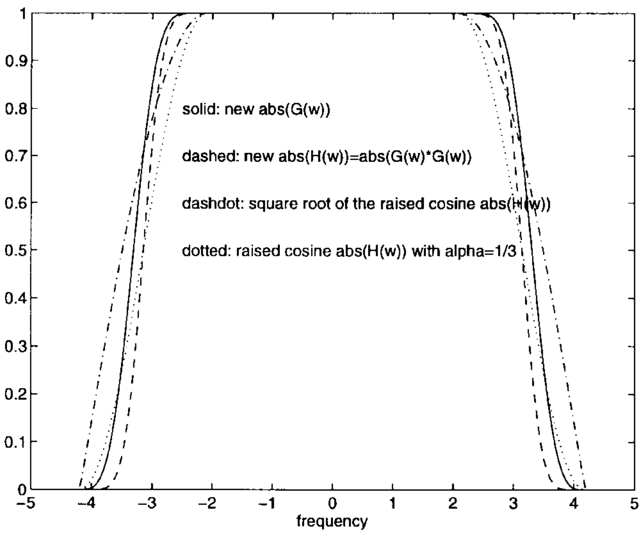


Fig. 1. The frequency responses $|H(\omega)|$ and $|G(\omega)|$ for the new pulse shaping and the raised cosine filters with $\alpha = 1/3$.

## III. Conclusion

In this letter, we proposed a new family of pulse-shaping filters. These pulse-shaping filters are ISI free with or without matched filtering at the receiver. This property may reduce the hardware cost in designing modem systems in some applications where the low-pass (bandpass) filtering is performed before the matched filtering. It should be noticed that, although the new pulse-shaping filters are real valued, they are not linear phase.

## Acknowledgment

The author would like to thank Dr. C.-M. Lo at Texas Instruments Incorporated for various discussions on modem design issues. He also wishes to thank the referees for their detailed comments that have improved the clarity of this paper.

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