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A method with error estimates for band-limited signal extrapolation from inaccurate data

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Abstract. In this paper, we consider the problem of extrapolation of a band-limited signal outside a fixed interval from its (approximate or contaminated) values in that interval. We propose a new extrapolation method that estimates the error between the extrapolated and true values, and which also resolves the ill-posedness of the problem. The method is called a modified minimum norm solution (MMNS) method. Both the continuous MMNS and its discretization are studied. The error estimates hold for some classes of band-limited signals, when the maximum magnitude of the data error is known. These classes of band-limited signals are also characterized.

1. Introduction

Let f be a finite energy signal, i.e. $f \in L^2(\mathbb{R})$. Its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}t\omega}\,\mathrm{d}t. \tag{1.1}$$

If there exists a positive number Ω such that $\hat{f}(\omega) = 0$ when $|\omega| > \Omega$, f is called Ω *band limited*. An Ω band-limited signal f can be represented by its inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}\omega. \tag{1.2}$$

It is known (see for example [1]) that a band-limited signal f is the restriction to the real line $\mathbb{R}$ of an entire function defined on the complex plane $\mathbb{C}$. Therefore, in theory, f is determined everywhere by its values on an interval no matter how small this interval is. This motivates the following band-limited signal extrapolation problem.

How does one practically extrapolate an Ω band-limited signal f outside an interval $[-T, T]$ when $f(t)$ is given for $t \in [-T, T]$ with a certain contamination error?

The above extrapolation problem is interesting not only in theory but also in many applications, such as spectral estimation (Papoulis [25]) and limited-angle tomography in medical image reconstruction (Natterer [24]), where only limited observation data are available.

Since f is analytic, a trivial solution for the problem is to compute the derivatives $f^{(n)}$ at $t = 0$ by using the values of f in $[-T, T]$ and then use the Taylor expansion. However,

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this method is extremely unstable due to the instability of the derivative computations. Numerical differentiation is an ill-posed problem and the degree of ill-posedness (which can be made precise using Sobolev negative norms) increases with the order of differentiation. Therefore, researchers have been seeking other methods. Since the early 1970s there has been considerable interest in this area, for example [4–8, 11–17, 24–30, 32–36, 38-40]. Since the problem itself is basically an inverse problem, it has been recognized that the existing extrapolation methods are generally unstable in terms of inaccurate data. The extrapolated values can change dramatically when the given data in an interval change slightly, see for example [27]. There are also many modified algorithms that have been proposed to improve the extrapolation performance. However, to the best of our knowledge there is no extrapolation algorithm with which one is able to estimate the error between the extrapolated and true values outside the given interval $[-T, T]$ for any nontrivial class of Ω band-limited signals, when the given data are inaccurate.

In this paper, we propose a new extrapolation method for band-limited signals that we call a *modified minimum norm solution* (MMNS) method. With the MMNS method we are able to estimate the error between the extrapolated and true values for some nontrivial classes of band-limited signals, when the maximum magnitude of the error of the given inaccurate data in a certain interval is known. This paper is organized as follows. In section 2 we study the MMNS method for continuous-time signals. In section 3 we study the MMNS method for discrete-time signals, which is a discretization of the method in section 2. In section 4 we present tractable characterizations of the classes of band-limited signals studied in sections 2 and 3. In section 5 we make several remarks.

2. Band-limited signal extrapolation in the continuous-time domain

In this section, we study the MMNS method for continuous-time band-limited signals. Without loss of generality, in what follows we assume $\Omega = 2\pi$ and $T = 1$ although we continue to use Ω and T to emphasize where they appear. We also assume $f_\epsilon = f + \eta$ where η is the error signal that is continuous in time and $|\eta(t)| \leqslant \epsilon$ for $t \in [-T, T]$, and $f_\epsilon(t)$ for $t \in [-T, T]$ are the given data. By normalization, we may assume that the maximal error magnitude $\epsilon < 1$.

We first introduce some notation. Let $L^2[-D, D]$ denote the space of all signals f that satisfy

$$\|f\|_{(D)} \triangleq \left(\int_{-D}^{D} |f(t)|^2 \, \mathrm{d}t \right)^{1/2} < \infty$$

where D is a positive number or ∞.

Let $\mathcal{BL}$ denote all Ω band-limited signals. For $\gamma \geqslant 0$, let $\mathcal{BL}_\gamma$ denote all Ω band-limited signals $f \in \mathcal{BL}$ that satisfy the following condition.

For any $\delta > 0$, there exists a signal $g_\delta \in L^2[-T, T]$ such that

$$\hat{f}_\delta(\omega) \triangleq \frac{1}{2\pi} \int_{-T}^{T} g_\delta(t) \mathrm{e}^{\mathrm{i}t\omega} \, \mathrm{d}t \tag{2.1}$$

satisfies the following two properties:

$$\|\hat{f} - \hat{f}_\delta\|_{(\Omega)} \leqslant \delta \tag{2.2}$$

and

$$\|\hat{f}_\delta\|_{(\infty)} \leqslant C\delta^{-\gamma} \tag{2.3}$$

where C is a constant that is independent of δ and γ, and $\hat{f}$ is the Fourier transform of f.

The physical meaning of the above subspace of all Ω band-limited signals is as follows. For an Ω band-limited signal f, its Fourier transform $\hat{f}$ is supported in $[-\Omega, \Omega]$ and $\hat{f} \in L^2[-\Omega, \Omega]$. The correspondence between the space $\mathcal{BL}$ of all Ω band-limited signals and the space $L^2[-\Omega, \Omega]$ of all finite L^2 norm signals defined on $[-\Omega, \Omega]$ is one-to-one and onto. Therefore, for a general Ω band-limited signal f its Fourier transform $\hat{f}$ may not have any smoothness property. The subspace $\mathcal{BL}_\gamma$ contains all Ω band-limited signals f with the following properties.

(i) The Fourier transform $\hat{f}$ can be approximated in the L^2 sense by a family $\{\hat{f}_\delta\}$ of T band-limited signals (entire functions of exponential order). This approximation holds inside the frequency band of f, i.e. the support $[-\Omega, \Omega]$ of $\hat{f}$.

(ii) The L^2 norms on the whole real line of the signals in the family $\{\hat{f}_\delta\}$ may not be uniformly bounded, but the rate of the *divergence* is not arbitrary. Rather the rate is related to the rate of the *convergence* of $\{\hat{f}_\delta\}$ in $L^2[-\Omega, \Omega]$ to $\hat{f}$ as $\delta \to 0$.

In this approximations framework, what is gained is the smoothness while what is lost is the boundedness of the family of L^2 norms on the real line. This trade-off is similar to the bandwidth and the timewidth trade-off [29, 30]. More precise interpretation and characterization of the above subspace will be given in section 4.

For the maximal error magnitude ϵ mentioned at the beginning of this section and any number $\lambda \geqslant 0$, let $\mathcal{BT}_{\epsilon,\lambda}$ denote the set of all signals $g \in L^2[-T, T]$ such that

$$\left| \frac{1}{2\pi^2} \int_{-T}^{T} \frac{\sin 2\pi(s-t)}{s-t} g(s)\,\mathrm{d}s - f_\epsilon(t) \right| \leqslant \lambda \qquad \text{for } t \in [-T, T]. \tag{2.4}$$

The basic idea for this subspace is to find signals in a neighbourhood of the inaccurate data signal $f_\epsilon(t)$ for $t \in [-T, T]$ such that the Fourier transforms of these signals are T band limited.

For $\lambda > \epsilon$, let $g_{\epsilon,\lambda}$ be the unique element (the existence and uniqueness will be shown in lemma 2) in $\mathcal{BT}_{\epsilon,\lambda}$ that has the minimum norm:

$$\|g_{\epsilon,\lambda}\|_{(T)} = \min\{\|g\|_{(T)};\, g \in \mathcal{BT}_{\epsilon,\lambda}\}. \tag{2.5}$$

Let

$$f_{\epsilon,\lambda}(t) = \frac{1}{2\pi^2} \int_{-T}^{T} \frac{\sin 2\pi(s-t)}{s-t} g_{\epsilon,\lambda}(s)\,\mathrm{d}s \tag{2.6}$$

which is called the MMNS of the continuous-time band-limited signal extrapolation problem. We now have the following error analysis for the above MMNS.

Theorem 1. *Let $f_{\epsilon,2\epsilon}$ be defined by (2.6) with the constant $\lambda = 2\epsilon$. If $f \in \mathcal{BL}_\gamma$ for some number γ with $0 \leqslant \gamma < \frac{1}{2}$, then*

$$|f_{\epsilon,2\epsilon}(t) - f(t)| \leqslant C\epsilon^{(1-2\gamma)/3} \qquad \textit{for all } t \in \mathbb{R} \tag{2.7}$$

where C is a constant independent of ϵ and γ.

Before we prove theorem 1, we establish two lemmas. We first recall the following known results from operator theory of ill-posed problems. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Hilbert spaces, and K be a bounded linear operator from $\mathbb{H}_1$ to $\mathbb{H}_2$. Let K^* denote the adjoint of the operator K and $K^\dagger$ be the generalized inverse of K (see [9, 19, 20]). Let $\mathcal{R}(K^*)$ denote the range of the operator K^*.

We recall that the (Moore–Penrose) generalized inverse $K^\dagger$ of the operator K is characterized by the following extremal property. For any g in the domain $\mathcal{D}(K^\dagger) = \mathcal{R}(K) + \mathcal{R}(K)^\perp$, the element $K^\dagger g$ is the minimal norm least-squares solution of the operator equation $Kf = g$. If $\mathcal{R}(K)$ is nonclosed, which is the case, for example, when K is a compact operator with infinite-dimensional range, then the operator $K^\dagger$ is unbounded, so the problem is ill-posed. The well known Tikhonov regularization uses the approximation

$$x_\alpha = (K^*K + \alpha I)^{-1}K^*g \qquad \alpha > 0$$

where I is the identity operator. It is well known that

$$\lim_{\alpha\to 0} x_\alpha = K^\dagger g \qquad \text{for } g \in \mathcal{D}(K^\dagger).$$

Without any 'smoothness' assumption on $K^\dagger g$, it is not possible in general to estimate the rate of convergence of x_α to $K^\dagger g$ or to obtain an error estimate $\|x_\alpha - K^\dagger g\|$ for fixed $\alpha > 0$. In what follows we will use the following proposition (see, e.g., [10, 18]) which states that if $K^\dagger g \in \mathcal{R}(K^*)$, a kind of smoothness condition, then an error estimate holds.

Proposition 1. *If $K^\dagger g \in \mathcal{R}(K^*)$, say $K^\dagger g = K^*g^*$ for some $g^* \in \mathbb{H}_2$, then*

$$\|K^\dagger g - x_\alpha\| \leqslant \sqrt{\alpha}\|g^*\|.$$

Let us consider the operator F^{-1} from $L^2[-\Omega,\Omega]$ to $L^2[-T,T]$, a restriction of the inverse Fourier transform (1.2), defined by:

$$(F^{-1}\hat{f})(t) = f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}t\omega}\mathrm{d}\omega \qquad t \in [-T,T]. \tag{2.8}$$

Then its adjoint $(F^{-1})^*$ is

$$[(F^{-1})^*g](\omega) = \frac{1}{2\pi}\int_{-T}^{T} g(s)\mathrm{e}^{\mathrm{i}s\omega}\,\mathrm{d}s \qquad \omega \in [-\Omega,\Omega].$$

From (2.8), $(F^{-1}\hat{f})(t) = 0$ for almost all $t \in [-T,T]$ if and only if $\hat{f}(\omega) = 0$ for almost all $\omega \in [-\Omega,\Omega]$. This implies that the null space $\mathcal{N}(F^{-1})$ of the operator F^{-1} is the zero element. This also implies that the space $\mathcal{R}((F^{-1})^*)$ is dense in $L^2[-\Omega,\Omega]$ since Closure$(\mathcal{R}((F^{-1})^*)) = \mathcal{N}(F^{-1})^\perp = L^2[-\Omega,\Omega]$. Thus we have proved the following lemma.

Lemma 1. *For any $\delta > 0$, there exists $g_\delta \in L^2[-T,T]$ such that*

$$\|\hat{f} - \hat{f}_\delta\|_{(\Omega)} \leqslant \delta$$

where

$$\hat{f}_\delta(\omega) = \frac{1}{2\pi}\int_{-T}^{T} g_\delta(s)\mathrm{e}^{\mathrm{i}s\omega}\,\mathrm{d}s$$

and $\hat{f}$ is the Fourier transform of f.

By lemma 1 and its implication in the time domain, it is clear that the set $\mathcal{BT}_{\epsilon,\lambda}$ defined by (2.4) is not empty when $\lambda > \epsilon$. Since the set $\mathcal{BT}_{\epsilon,\lambda}$ is closed and convex, we have proved the following.

Lemma 2. *For $\lambda > \epsilon$, there is a unique element $g_{\epsilon,\lambda}$ in $\mathcal{BT}_{\epsilon,\lambda}$ such that*

$$\|g_{\epsilon,\lambda}\|_{(T)} = \min\{\|g\|_{(T)} : g \in \mathcal{BT}_{\epsilon,\lambda}\}.$$

With the function $g_{\epsilon,\lambda}$ as in lemma 2, define

$$\bar{g}_{\epsilon,\lambda}(\omega) = \frac{1}{2\pi}\int_{-T}^{T} g_{\epsilon,\lambda}(s)\mathrm{e}^{\mathrm{i}s\omega}\,\mathrm{d}s. \tag{2.9}$$

Then the MMNS $f_{\epsilon,\lambda}$ in (2.6) can also be represented as

$$f_{\epsilon,\lambda}(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \bar{g}_{\epsilon,\lambda}(\omega)\mathrm{e}^{-\mathrm{i}s\omega}\mathrm{d}\omega.$$

With the signal $\hat{f}_\delta$ in (2.1), define

$$\bar{f}_\delta(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}_\delta(s)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}s. \tag{2.10}$$

We are now ready to prove theorem 1.

Proof of theorem 1. When $f \in \mathcal{BL}_\gamma$ for $\gamma \geqslant 0$, by (2.1), (2.2) the signal g_δ with $\delta = (2\pi/\sqrt{2\Omega})\epsilon$ satisfies

$$\|\hat{f} - \hat{f}_\delta\|_{(\Omega)} \leqslant \frac{2\pi}{\sqrt{2\Omega}}\epsilon$$

where $\hat{f}_\delta$ is related to g_δ via (2.1). In the time domain, by using the Cauchy–Schwarz inequality and the above inequality we have

$$|f(t) - f_\delta(t)| \leqslant \frac{1}{2\pi}\left|\int_{-\Omega}^{\Omega}(\hat{f}(\omega) - \hat{f}_\delta(\omega))\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}\omega\right| \leqslant \epsilon$$

where

$$\begin{aligned} f_\delta(t) &= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}_\delta(\omega)\mathrm{e}^{-\mathrm{i}t\omega}\mathrm{d}\omega \\ &\overset{(2.1)}{=} \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\frac{1}{2\pi}\int_{-T}^{T} g_\delta(s)\mathrm{e}^{\mathrm{i}\omega(s-t)}\mathrm{d}s\,\mathrm{d}\omega \\ &= \frac{1}{2\pi^2}\int_{-T}^{T}\frac{\sin 2\pi(s-t)}{s-t}g_\delta(s)\,\mathrm{d}s \end{aligned}$$

where the convention $\Omega = 2\pi$ made at the beginning of this section is used. By the assumption

$$|f_\epsilon(t) - f(t)| \leqslant \epsilon$$

we have

$$|f_\delta(t) - f_\epsilon(t)| \leqslant 2\epsilon.$$

According to (2.4), we have proved that g_δ is in $\mathcal{BT}_{\epsilon,2\epsilon}$. Hence, by lemma 2 we obtain

$$\|g_{\epsilon,2\epsilon}\|_{(T)} \leqslant \|g_{(2\pi/\sqrt{2\Omega})\epsilon}\|_{(T)}.$$

Moreover, by (2.1) and (2.3), we have

$$\|g_{\epsilon,2\epsilon}\|_{(T)} \leqslant \|g_{(2\pi/\sqrt{2\Omega})\epsilon}\|_{(T)} \leqslant 2\pi C(2\pi/\sqrt{2\Omega})^{-\gamma}\epsilon^{-\gamma}.$$

Since

$$|f_{\epsilon,2\epsilon}(t) - f_\epsilon(t)| \leqslant 2\epsilon \qquad t \in [-T, T]$$

we have

$$|f_{\epsilon,2\epsilon}(t) - f(t)| \leqslant 3\epsilon \qquad t \in [-T, T].$$

For the signal $\bar{f}_\delta$ in (2.10) and considering (2.2) in the time domain, we have

$$|\bar{f}_\delta(t) - f(t)| \leqslant \frac{\sqrt{2\Omega}}{2\pi}\delta \qquad \text{for } t \in \mathbb{R}.$$

Therefore,

$$|f_{\epsilon,2\epsilon}(t) - \bar{f}_\delta(t)| \leqslant 3\epsilon + \frac{\sqrt{2\Omega}}{2\pi}\delta \qquad \text{for } t \in [-T, T]. \tag{2.11}$$

For $\alpha > 0$, let

$$x_\alpha = ((F^{-1})^* F^{-1} + \alpha I)^{-1} (F^{-1})^* (f_{\epsilon,2\epsilon}(t) - \bar{f}_\delta(t)).$$

By using proposition 1 with $K = F^{-1}$ and $\delta = \epsilon$, and (2.1), (2.2), we have

$$\begin{aligned}\|\bar{g}_{\epsilon,2\epsilon} - \hat{f}_\delta - x_\alpha\|_{(\Omega)} &= \|K^\dagger(f_{\epsilon,2\epsilon} - \bar{f}_\delta) - x_\alpha\|_{(\Omega)} \\ &\leqslant \sqrt{\alpha}(\|g_{\epsilon,2\epsilon}\|_{(T)} + \|g_\delta\|_{(T)}) \\ &\leqslant 2\pi C \epsilon^{-\gamma}\sqrt{\alpha},\end{aligned}$$

where C is a constant, and $\bar{g}_{\epsilon,2\epsilon} - \hat{f}_\delta = K^*(g_{\epsilon,2\epsilon} - g_\delta)$ from (2.1) and (2.9). On the other hand,

$$\|x_\alpha\|_{(\Omega)} \leqslant \frac{1}{\alpha}\frac{T\sqrt{2\Omega}}{\pi}\left(3\epsilon + \frac{\sqrt{2\Omega}}{2\pi}\delta\right) = \frac{T\sqrt{2\Omega}}{\pi}\left(3 + \frac{\sqrt{2\Omega}}{2\pi}\right)\frac{\epsilon}{\alpha}.$$

Thus,

$$\|\bar{g}_{\epsilon,2\epsilon} - \hat{f}_\delta\|_{(\Omega)} \leqslant 2\pi C\epsilon^{-\gamma}\sqrt{\alpha} + \frac{T\sqrt{2\Omega}}{\pi}\left(3 + \frac{\sqrt{2\Omega}}{2\pi}\right)\frac{\epsilon}{\alpha}.$$

Using (2.2) with $\delta = \epsilon$, we have

$$\|\bar{g}_{\epsilon,2\epsilon} - \hat{f}\|_{(\Omega)} \leqslant 2\pi C\epsilon^{-\gamma}\sqrt{\alpha} + \frac{T\sqrt{2\Omega}}{\pi}\left(3 + \frac{\sqrt{2\Omega}}{2\pi}\right)\frac{\epsilon}{\alpha} + \epsilon.$$

In the time domain, using the Cauchy–Schwarz inequality, we obtain

$$|f_{\epsilon,2\epsilon}(t) - f(t)| \leqslant \frac{\sqrt{2\Omega}}{2\pi}\left[2\pi C\epsilon^{-\gamma}\sqrt{\alpha} + \frac{T\sqrt{2\Omega}}{\pi}\left(3 + \frac{\sqrt{2\Omega}}{2\pi}\right)\frac{\epsilon}{\alpha} + \epsilon\right] \qquad \text{for } t \in \mathbb{R}.$$

Therefore, estimate (2.7) in theorem 1 can be proved by taking $\alpha = \epsilon^{2(1+\gamma)/3}$ and using the assumption $\epsilon < 1$ made at the beginning of this section. □

3. Discretization of the MMNS method

Since in practice we usually process discrete-time signals, it is very important to consider the discretization of the MMNS method proposed in section 2. To do so, we need some notation.

For any number λ with $\lambda > \epsilon$ and positive integer m, let $\mathcal{M}l^2_\lambda(2m+1)$ denote the set of $(2m+1)$-dimensional vectors $\boldsymbol{a} = \{a(k)\} \in \mathbb{C}^{2m+1}$ such that

$$\left|\frac{1}{2\pi^2}\frac{1}{m}\sum_{k=-m}^{m}\frac{\sin 2\pi\left(\frac{k}{m} - \frac{n}{m}\right)}{\frac{k}{m} - \frac{n}{m}}a(k) - f_\epsilon\left(\frac{n}{m}\right)\right| \leqslant \lambda \qquad \text{for } -m \leqslant n \leqslant m. \tag{3.1}$$

For $\lambda > \epsilon$, let $\boldsymbol{z}_m^\lambda = \{z_m^\lambda(k)\}$ be the unique element (the existence and the uniqueness will be shown in lemma 4) in $\mathcal{M}l_\lambda(2m+1)$ such that

$$\|\boldsymbol{z}_m^\lambda\| = \min\{\|\boldsymbol{a}\|; \boldsymbol{a} = \{a(k)\} \in \mathcal{M}l_\lambda^2(2m+1)\} \tag{3.2}$$

where

$$\|\boldsymbol{a}\| \triangleq \left(\sum_{k=-m}^{m} |a(k)|^2\right)^{1/2}.$$

Finally, let

$$\Psi_m^\lambda(t) = \frac{1}{2\pi^2}\frac{1}{m}\sum_{k=-m}^{m}\frac{\sin 2\pi\left(\frac{k}{m}-t\right)}{\frac{k}{m}-t} z_m^\lambda(k). \tag{3.3}$$

Notice that, for a signal $f \in \mathcal{BL}$ and any constants $\lambda > \epsilon \geqslant 0$ and any positive integer m, we can always construct the signal $f_{\epsilon,\lambda}$ in (2.6) and the signal Ψ_m^λ in (3.3) from the given data $f_\epsilon(t)$ for $t \in [-T, T]$. In other words, the MMNS $f_{\epsilon,\lambda}$ given in (2.6) and its discretization Ψ_m^λ in (3.3) can be found for any $f \in \mathcal{BL}$ using its known values on a segment.

In practice, it is usually difficult to get the MMNS $f_{\epsilon,\lambda}$ in (2.6). A practical way to compute it is to use the discretization form that is formulated by Ψ_m^λ in (3.3). We have the following convergence of the discretization Ψ_m^λ of the MMNS.

Theorem 2. *For any constant λ with $\lambda > \epsilon$, the discretization Ψ_m^λ converges to $f_{\epsilon,\lambda}$ uniformly on compact sets of $\mathbb{R}$ when $m \to \infty$.*

It is interesting to notice that the convergence result in theorem 2 does not require any additional condition for a band-limited signal f. In order to get an error estimation for the MMNS, an additional condition, i.e. $f \in \mathcal{BL}_\gamma$, in theorem 1 is needed.

To prove theorem 2, we need several lemmas.

Lemma 3. *For each fixed $\lambda_0 > \epsilon$, there exists $M > 0$ such that, when $m > M$ and $\lambda \geqslant \lambda_0$, the set $\mathcal{M}l_\lambda^2(2m+1)$ defined in (3.1) is not empty and $\|\boldsymbol{z}_m^\lambda\| \leqslant C_{\lambda_0}$, where C_{λ_0} is some positive constant and independent of m and λ with $\lambda \geqslant \lambda_0$.*

Proof. By lemma 1, for $\delta = (\lambda - \epsilon)/3$, there exists $g_\delta \in L^2[-1, 1]$ such that

$$\|\hat{f} - \hat{f}_\delta\|_{(2\pi)} \leqslant (\lambda - \epsilon)/3$$

where

$$\hat{f}_\delta(\omega) = \frac{1}{2\pi}\int_{-1}^{1} g_\delta(s)\mathrm{e}^{\mathrm{i}s\omega}\,\mathrm{d}s.$$

Thus,

$$\left|\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\hat{f}_\delta(\omega)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}\omega - f(t)\right| \leqslant (\lambda - \epsilon)/(3\sqrt{\pi}) \qquad \text{for all } t \in \mathbb{R}.$$

In other words,

$$\left|\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\mathrm{e}^{-\mathrm{i}t\omega}\frac{1}{2\pi}\int_{-1}^{1} g_\delta(s)\mathrm{e}^{\mathrm{i}s\omega}\mathrm{d}s\mathrm{d}\omega - f(t)\right| \leqslant (\lambda - \epsilon)/3 \qquad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

Since the space of continuous functions is dense in $L^2[-1, 1]$, there exists $h_\delta \in C[-1, 1]$ such that

$$\|g_\delta - h_\delta\|_{(1)} \leqslant (\lambda - \epsilon)/3.$$

Thus,

$$\left|\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\mathrm{e}^{-\mathrm{i}t\omega}\frac{1}{2\pi}\int_{-1}^{1}(g_\delta(s)-h_\delta(s))\mathrm{e}^{\mathrm{i}s\omega}\,\mathrm{d}s\,\mathrm{d}\omega\right| \leqslant \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\frac{1}{2\pi}\int_{-1}^{1}|g_\delta(s)-h_\delta(s)|\,\mathrm{d}s\,\mathrm{d}\omega$$
$$\leqslant \sqrt{2}(\lambda-\epsilon)/(3\pi) < (\lambda-\epsilon)/3 \qquad \text{for all } t\in\mathbb{R}.$$

By (3.4) we have

$$\left|\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\mathrm{e}^{-\mathrm{i}t\omega}\frac{1}{2\pi}\int_{-1}^{1}h_\delta(s)\mathrm{e}^{\mathrm{i}s\omega}\mathrm{d}s\mathrm{d}\omega - f_\epsilon(t)\right| \leqslant (2\lambda+\epsilon)/3 \qquad \text{for all } t\in[-1,1].$$

That is,

$$\left|\frac{1}{2\pi^2}\int_{-1}^{1}\frac{\sin 2\pi(s-t)}{s-t}h_\delta(s)\mathrm{d}s - f_\epsilon(t)\right| \leqslant (2\lambda+\epsilon)/3 \qquad \text{for all } t\in[-1,1]. \tag{3.5}$$

Since h_δ is continuous on $[-1,1]$, the following sum

$$\frac{1}{2\pi^2}\frac{1}{m}\sum_{k=-m}^{m}\frac{\sin 2\pi(\frac{k}{m}-t)}{\frac{k}{m}-t}h_\delta\left(\frac{k}{m}\right)$$

converges uniformly to

$$\frac{1}{2\pi^2}\int_{-1}^{1}\frac{\sin 2\pi(s-t)}{s-t}h_\delta(s)\,\mathrm{d}s$$

for $t\in[-1,1]$. Therefore, for $(\lambda-\epsilon)/3$, there exists $M>0$ such that, when $m>M$, we have

$$\left|\frac{1}{2\pi^2}\frac{1}{m}\sum_{k=-m}^{m}\frac{\sin 2\pi\left(\frac{k}{m}-\frac{n}{m}\right)}{\frac{k}{m}-\frac{n}{m}}h_\delta\left(\frac{k}{m}\right) - \frac{1}{2\pi^2}\int_{-1}^{1}\frac{\sin 2\pi(s-\frac{n}{m})}{s-\frac{n}{m}}h_\delta(s)\,\mathrm{d}s\right|$$
$$\leqslant (\lambda-\epsilon)/3 \qquad \text{for } |n|\leqslant m.$$

Combining this with (3.5), we obtain

$$\left|\frac{1}{2\pi^2}\frac{1}{m}\sum_{k=-m}^{m}\frac{\sin 2\pi\left(\frac{k}{m}-\frac{n}{m}\right)}{\frac{k}{m}-\frac{n}{m}}h_\delta\left(\frac{k}{m}\right) - f_\epsilon\left(\frac{n}{m}\right)\right| \leqslant \lambda \qquad \text{for all } |n|\leqslant m.$$

Let $a(k) = h_\delta(\frac{k}{m})$ for $|k|\leqslant m$. Then, $\{a(k)\}\in\mathcal{M}l_\lambda^2(2m+1)$. This proves that the set $\mathcal{M}l_\lambda^2(2m+1)$ is not empty when $m>M$.

Moreover, the above M can be large enough such that, when $m>M$,

$$\begin{aligned}\frac{1}{m}\sum_{k=-m}^{m}|a(k)|^2 &\leqslant \int_{-1}^{1}|h_\delta(s)|^2\,\mathrm{d}s+1\\ &= \int_{-1}^{1}|h_{(\lambda-\epsilon)/3}(s)|^2\,\mathrm{d}s+1\\ &\leqslant (\|g_{(\lambda-\epsilon)/3}\|_{(1)}+(\lambda-\epsilon)/3)^2+1.\end{aligned}$$

Let $(\lambda-\epsilon)/3<1$ and

$$C_{\lambda_0} = \left\{(\|g_{(\lambda_0-\epsilon)/3}\|_{(1)}+1)^2+1\right\}^{1/2}.$$

Then lemma 3 is proved. □

Similar to lemma 2, since the set $\mathcal{M}l^2_\lambda(2M+1)$ is closed and convex, we prove lemma 4.

Lemma 4. *For every m and λ with $\lambda > \epsilon$, there exists a unique element*

$$\boldsymbol{z}_m^\lambda = \{z_m^\lambda(k)\} \in \mathcal{M}l^2_\lambda(2m+1)$$

such that

$$\|\boldsymbol{z}_m^\lambda\| = \min\{\|\boldsymbol{a}\| : \boldsymbol{a} = \{a(k)\} \in \mathcal{M}l^2_\lambda(2m+1)\}.$$

Recall that a family of functions of a complex variable is called a *normal family* if every sequence of the family contains a subsequence which converges uniformly on compact sets. It is known that a family of functions that is uniformly bounded in any compact set is a normal family. We use this result in the proof of the following lemma.

Lemma 5. *For each λ_0 ($> \epsilon$), the family of functions $\{\Psi_m^\lambda(t)\}_{\lambda\geqslant\lambda_0,m}$ defined in (3.3) is normal when t is extended to the complex plane $\mathbb{C}$.*

Proof. The functions Ψ_m^λ in (3.3) can be rewritten as

$$\Psi_m^\lambda(t) = \frac{1}{4\pi^2}\frac{1}{m}\int_{-2\pi}^{2\pi} \mathrm{e}^{-\mathrm{i}t\omega}\sum_{k=-m}^{m}\mathrm{e}^{\mathrm{i}k\omega/m}z_m^\lambda(k)\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\mathrm{e}^{-\mathrm{i}t\omega}\left(\frac{1}{2\pi}\frac{1}{m}\sum_{k=-m}^{m}\mathrm{e}^{\mathrm{i}k\omega/m}z_m^\lambda(k)\right)\mathrm{d}\omega.$$

Thus,

$$|\Psi_m^\lambda(z)| \leqslant \frac{\mathrm{e}^{2\pi|z|}}{4\pi^2}\int_{-2\pi}^{2\pi}\left|\frac{1}{m}\sum_{k=-m}^{m}\mathrm{e}^{-\mathrm{i}k\omega/m}z_m^\lambda(k)\right|\mathrm{d}\omega$$
$$\leqslant \frac{\mathrm{e}^{2\pi|z|}}{\pi}\frac{1}{m}\sum_{k=-m}^{m}|z_m^\lambda(k)|$$
$$\leqslant \frac{\mathrm{e}^{2\pi|z|}}{\pi}\left(\frac{2m+1}{m}\right)^{1/2}\|\boldsymbol{z}_m^\lambda\|$$
$$\overset{\text{lemma 3}}{\leqslant} \frac{1}{\pi}\left(\frac{2m+1}{m}\right)^{1/2}C_{\lambda_0}\mathrm{e}^{2\pi|z|} \qquad \text{for } \lambda \geqslant \lambda_0 \qquad z \in \mathbb{C}.$$

This proves that the family $\{\Psi_m^\lambda\}_{\lambda\geqslant\lambda_0,m}$ is normal. □

Define

$$\phi_m^\lambda(\omega) = \frac{1}{2\pi}\frac{1}{m}\sum_{k=-m}^{m}\mathrm{e}^{\mathrm{i}k\omega/m}z_m^\lambda(k). \tag{3.6}$$

Lemma 6. *For each λ_0 ($> \epsilon$) the family $\{\phi_m^\lambda(z)\}_{\lambda\geqslant\lambda_0,m}$ is normal and its limit functions are* 1 *band limited.*

Proof. The proof of normality is similar to the proof of lemma 5 by using lemma 3. By Fatou's lemma and lemma 3, it is easy to prove that all limit functions of the family $\{\phi_m^\lambda(z)\}_{\lambda\geqslant\lambda_0,m}$ are in $L^2(\mathbb{R})$ when z is restricted to the real line $\mathbb{R}$. Therefore, by the Paley–Wiener theorem (see [1]), lemma 6 is proved. □

Lemma 7. *Let $g_{\epsilon,\lambda}$ be as defined in (2.5). For a fixed ϵ, let $h(\lambda) = \|g_{\epsilon,\lambda}\|_{(1)}$. Then the function $h(\lambda)$ is continuous for $\lambda > \epsilon$.*

Proof. Let λ_0 and λ_1 be any two positive numbers such that $\lambda_0 > \lambda_1 > \epsilon$. For any $\lambda \geqslant \lambda_1$, define

$$\bar{g}_{\epsilon,\lambda}(\omega) = \frac{1}{2\pi}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}s\omega} g_{\epsilon,\lambda}(s)\,\mathrm{d}s.$$

Then

$$|\bar{g}_{\epsilon,\lambda}(\omega)| = \left|\frac{1}{2\pi}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}s\omega} g_{\epsilon,\lambda}(s)\mathrm{d}s\right| \leqslant \mathrm{e}^{|\omega|}\frac{\sqrt{2}}{2\pi}\|g_{\epsilon,\lambda}\|_{(1)} \leqslant \frac{\sqrt{2}}{2\pi}\mathrm{e}^{|\omega|}\|g_{\epsilon,\lambda_1}\|_{(1)} \qquad \text{for } \lambda \geqslant \lambda_1.$$

This implies that the family $\{\bar{g}_{\epsilon,\lambda}(\omega)\}_{\lambda\geqslant\lambda_1}$ is normal. Similar to lemma 6, its limit functions are 1 band limited. Let $\bar{h}_{\epsilon,\lambda_0}$ be one of its limit functions. Let $\lambda(n) \to \lambda_0^+$ and suppose that the sequence $\{\bar{g}_{\epsilon,\lambda(n)}\}$ converges to $\bar{h}_{\epsilon,\lambda_0}$ uniformly on compact sets of $\mathbb{C}$. Then, there exists $h_{\epsilon,\lambda_0} \in L^2[-1,1]$ such that

$$\bar{h}_{\epsilon,\lambda_0}(\omega) = \frac{1}{2\pi}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}s\omega} h_{\epsilon,\lambda_0}(s)\mathrm{d}s.$$

By the definition of $f_{\epsilon,\lambda(n)}$ we have

$$\left|\frac{1}{2\pi}\int_{-2\pi}^{2\pi} \mathrm{e}^{-\mathrm{i}t\omega}\bar{g}_{\epsilon,\lambda(n)}(\omega)\mathrm{d}\omega - f_\epsilon(t)\right| = |f_{\epsilon,\lambda(n)}(t) - f_\epsilon(t)| \leqslant \lambda(n) \qquad \text{for } t \in [-1,1].$$

Let $n \to \infty$ in the above inequality,

$$\left|\frac{1}{2\pi}\int_{-2\pi}^{2\pi} \mathrm{e}^{-\mathrm{i}t\omega}\frac{1}{2\pi}\int_{-1}^{1} h_{\epsilon,\lambda_0}(s)\mathrm{e}^{\mathrm{i}s\omega}\,\mathrm{d}\omega\,\mathrm{d}s - f_\epsilon(t)\right| \leqslant \lambda_0 \qquad \text{for } t \in [-1,1].$$

Thus, $h_{\epsilon,\lambda_0} \in \mathcal{BT}_{\epsilon,\lambda_0}$. Therefore,

$$\|h_{\epsilon,\lambda_0}\|_{(1)} \geqslant \|g_{\epsilon,\lambda_0}\|_{(1)}. \tag{3.7}$$

On the other hand, for any $B > 0$,

$$\begin{aligned}\int_{-B}^{B} |\bar{h}_{\epsilon,\lambda_0}(\omega)|^2\mathrm{d}\omega &= \lim_{n\to\infty}\int_{-B}^{B} |\bar{g}_{\epsilon,\lambda(n)}(\omega)|^2\mathrm{d}\omega \\ &\leqslant \overline{\lim}_{n\to\infty}\int_{-\infty}^{\infty} |\bar{g}_{\epsilon,\lambda(n)}(\omega)|^2\mathrm{d}\omega = \overline{\lim}_{n\to\infty}\|\bar{g}_{\epsilon,\lambda(n)}\|_{(\infty)}^2 \\ &= \frac{1}{2\pi}\overline{\lim}_{n\to\infty}\|g_{\epsilon,\lambda(n)}\|_{(1)}^2 \leqslant \frac{1}{2\pi}\|g_{\epsilon,\lambda_0}\|_{(1)}^2.\end{aligned}$$

Therefore,

$$\|\bar{h}_{\epsilon,\lambda_0}\|_{(\infty)} \leqslant \frac{1}{\sqrt{2\pi}}\|g_{\epsilon,\lambda_0}\|_{(1)}.$$

In other words,

$$\|h_{\epsilon,\lambda_0}\|_{(1)} \leqslant \|g_{\epsilon,\lambda_0}\|_{(1)}.$$

By (3.7) and lemma 2, we have proved that $h_{\epsilon,\lambda_0} = g_{\epsilon,\lambda_0}$. Therefore, we have proved

$$\lim_{\lambda\to\lambda_0^+} h(\lambda) = h(\lambda_0). \tag{3.8}$$

Now we want to prove that

$$\lim_{\lambda\to\lambda_0^-} h(\lambda) = h(\lambda_0). \tag{3.9}$$

Let λ_1 be any positive with $0 < \lambda_1 < \lambda_0$. Let $\{\lambda(n)\}$ be any sequence of numbers with $\lambda_1 \leqslant \lambda(n) \leqslant \lambda(n+1) \leqslant \lambda_0$ that converges to λ_0. Define

$$h_n(s) = \left(1 - \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\right) g_{\epsilon,\lambda_1}(s) + \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1} g_{\epsilon,\lambda_0}(s) \qquad \text{for } s \in [-1,1]. \tag{3.10}$$

Define

$$\bar{h}_n(t) = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \mathrm{e}^{-\mathrm{i}t\omega}\left(\frac{1}{2\pi}\int_{-1}^{1} h_n(s)\mathrm{e}^{\mathrm{i}s\omega}\mathrm{d}s\right)\mathrm{d}\omega.$$

Then

$$\begin{aligned}|\bar{h}_n(t) - f_\epsilon(t)| &= \left|\frac{1}{2\pi^2}\int_{-1}^{1}\frac{\sin 2\pi(s-t)}{s-t}h_n(s)\,\mathrm{d}s - f_\epsilon(t)\right| \\ &\leqslant \left(1 - \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\right)\left|\frac{1}{2\pi^2}\int_{-1}^{1}\frac{\sin 2\pi(s-t)}{s-t}g_{\epsilon,\lambda_1}(s)\,\mathrm{d}s - f_\epsilon(t)\right| \\ &\quad + \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\left|\frac{1}{2\pi^2}\int_{-1}^{1}\frac{\sin 2\pi(s-t)}{s-t}g_{\epsilon,\lambda_0}(s)\,\mathrm{d}s - f_\epsilon(t)\right| \\ &\leqslant \left(1 - \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\right)\lambda_1 + \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\lambda_0 = \lambda(n).\end{aligned}$$

This implies that $h_n \in \mathcal{BT}_{\epsilon,\lambda(n)}$.

From (3.10) we have

$$\|h_n\|_{(1)} \leqslant \left(1 - \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\right)\|g_{\epsilon,\lambda_1}\|_{(1)} + \frac{\lambda(n)-\lambda_1}{\lambda_0-\lambda_1}\|g_{\epsilon,\lambda_0}\|_{(1)}.$$

Letting $n \to \infty$ we obtain

$$\overline{\lim}_{n\to\infty}\|h_n\|_{(1)} \leqslant \|g_{\epsilon,\lambda_0}\|_{(1)}.$$

Since we have proved that $h_n \in \mathcal{BT}_{\epsilon,\lambda(n)}$,

$$\|g_{\epsilon,\lambda(n)}\|_{(1)} \leqslant \|h_n\|_{(1)}.$$

This proves that

$$\overline{\lim}_{n\to\infty}\|g_{\epsilon,\lambda(n)}\|_{(1)} \leqslant \|g_{\epsilon,\lambda_0}\|_{(1)}.$$

On the other hand, the following is clear:

$$\|g_{\epsilon,\lambda(n)}\|_{(1)} \geqslant \|g_{\epsilon,\lambda_0}\|.$$

Thus,

$$\lim_{n\to\infty}\|g_{\epsilon,\lambda(n)}\|_{(1)} = \|g_{\epsilon,\lambda_0}\|_{(1)}$$

that is, (3.9) is proved. This proves lemma 7. □

We are now ready to prove theorem 2.

Proof of theorem 2. By (3.3) and (3.6) we have

$$\Psi_m^\lambda(t) = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\phi_m^\lambda(\omega)\mathrm{e}^{-\mathrm{i}t\omega}\mathrm{d}\omega.$$

If we can prove that every limit function of the sequence $\{\Psi_m^\lambda\}$ is $f_{\epsilon,\lambda}$, theorem 2 is proved. Assume $\bar{h}_{\epsilon,\lambda}$ is a limit function of the sequence $\{\Psi_m^\lambda\}$. Without loss of generality, we may assume the sequence $\{\Psi_m^\lambda\}$ converges to $\bar{h}_{\epsilon,\lambda}$. Since the family $\{\Psi_m^\lambda\}$ for a fixed λ is normal by lemma 5, the convergence is uniform on compact sets of $\mathbb{C}$. By lemma 6, the family

$\{\phi_m^\lambda\}$ is also normal for a fixed λ. We may assume that the sequence $\{\phi_m^\lambda\}$ converges to $\hat{h}_{\epsilon,\lambda}$ uniformly on compact sets of $\mathbb{C}$ and

$$\bar{h}_{\epsilon,\lambda}(t) = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \hat{h}_{\epsilon,\lambda}(\omega)\mathrm{e}^{-\mathrm{i}t\omega}\mathrm{d}\omega.$$

By Lemma 6, there exists $\tilde{h}_{\epsilon,\lambda} \in L^2[-1,1]$ such that

$$\hat{h}_{\epsilon,\lambda}(\omega) = \frac{1}{2\pi}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}s\omega}\tilde{h}_{\epsilon,\lambda}(s)\mathrm{d}s.$$

Taking the limit as $m \to \infty$ in

$$\left|\Psi_m^\lambda\left(\frac{n}{m}\right) - f_\epsilon\left(\frac{n}{m}\right)\right| \leqslant \lambda \qquad \text{for } |n| \leqslant m$$

and using the continuity of $\bar{h}_{\epsilon,\lambda}(t)$ and $f_\epsilon(t)$ for $t \in [-1,1]$, we obtain

$$|\bar{h}_{\epsilon,\lambda}(t) - f_\epsilon(t)| \leqslant \lambda \qquad t \in [-1,1].$$

This proves that $\tilde{h}_{\epsilon,\lambda} \in \mathcal{BT}_{\epsilon,\lambda}$. Thus,

$$\|\tilde{h}_{\epsilon,\lambda}\|_{(1)} \geqslant \|g_{\epsilon,\lambda}\|_{(1)}. \tag{3.11}$$

We next want to prove the reverse inequality.

For $\lambda > \epsilon$, choose μ such that $\lambda > \mu > \epsilon$. For this μ, we have $g_{\epsilon,\mu} \in \mathcal{BT}_{\epsilon,\lambda}$. Using the same argument as in the proof of lemma 3, for $(\lambda-\mu)/3$ there exists $\tilde{g}_{\epsilon,\mu} \in C[-1,1]$ such that

$$\|g_{\epsilon,\mu} - \tilde{g}_{\epsilon,\mu}\|_{(1)} \leqslant \frac{\lambda-\mu}{3}.$$

Thus, if we let

$$\bar{\tilde{g}}_{\epsilon,\mu}(t) = \frac{1}{2\pi^2}\int_{-1}^{1} \frac{\sin 2\pi(s-t)}{s-t}\tilde{g}_{\epsilon,\mu}(s)\,\mathrm{d}s$$

then,

$$|\bar{\tilde{g}}_{\epsilon,\mu}(t) - f_{\epsilon,\mu}(t)| \leqslant \frac{\sqrt{2}}{\pi}\frac{\lambda-\mu}{3} \qquad \text{for } t \in [-1,1].$$

Therefore, there exists $M > 0$ such that when $m > M$ we have

$$\left|\frac{1}{2\pi^2}\frac{1}{m}\sum_{k=-m}^{m} \frac{\sin 2\pi\left(\frac{k}{m}-\frac{n}{m}\right)}{\frac{k}{m}-\frac{n}{m}}\tilde{g}_{\epsilon,\mu}\left(\frac{k}{m}\right) - f_\epsilon\left(\frac{n}{m}\right)\right| \leqslant \frac{2\lambda+\mu}{3} < \lambda.$$

By (3.1), this implies that $\tilde{\boldsymbol{g}}_{\epsilon,\mu} = \{\tilde{g}_{\epsilon,\mu}(\frac{k}{m})\} \in \mathcal{M}l_\lambda^2(2m+1)$. Therefore,

$$\|\tilde{\boldsymbol{g}}_{\epsilon,\mu}\| \geqslant \|\boldsymbol{z}_m^\lambda\|.$$

Thus

$$\overline{\lim}_{m\to\infty}\frac{1}{m}\|\boldsymbol{z}_m^\lambda\|^2 \leqslant \overline{\lim}_{m\to\infty}\frac{1}{m}\|\tilde{\boldsymbol{g}}_{\epsilon,\mu}\|^2 = \|\tilde{g}_{\epsilon,\mu}\|_{(1)}^2.$$

Therefore,

$$\left(\overline{\lim}_{m\to\infty}\frac{1}{m}\|\boldsymbol{z}_m^\lambda\|^2\right)^{1/2} \leqslant \|g_{\epsilon,\mu}\|_{(1)} + \frac{\lambda-\mu}{3}. \tag{3.12}$$

On the other hand, for any $B > 0$,

$$\begin{aligned}\int_{-B}^{B} |\hat{h}_{\epsilon,\lambda}(\omega)|^2 &= \lim_{m\to\infty} \int_{-B}^{B} |\phi_m^\lambda(\omega)|^2 \mathrm{d}\omega \\ &\leqslant \overline{\lim}_{m\to\infty} \int_{-\pi m}^{\pi m} |\phi_m^\lambda(\omega)|^2 \mathrm{d}\omega \\ &\overset{(3.6)}{=} \overline{\lim}_{m\to\infty} \int_{-\pi m}^{\pi m} \left(\frac{1}{2\pi m}\sum_{k=-m}^{m} \mathrm{e}^{\mathrm{i}k\omega/m} z_m^\lambda(k)\right)\left(\frac{1}{2\pi m}\sum_{k=-m}^{m} \mathrm{e}^{-\mathrm{i}k\omega/m} \overline{z_m^\lambda(k)}\right) \mathrm{d}\omega \\ &= \overline{\lim}_{m\to\infty} \int_{-\pi m}^{\pi m} \frac{1}{(2\pi)^2 m^2} |z_m^\lambda(k)|^2 \mathrm{d}\omega \\ &= \overline{\lim}_{m\to\infty} \frac{1}{2\pi m} \sum_{k=-m}^{m} |z_m^\lambda(k)|^2.\end{aligned}$$

Therefore,

$$\|\hat{h}_{\epsilon,\lambda}\|_{(\infty)}^2 \leqslant \frac{1}{2\pi} \overline{\lim}_{m\to\infty} \frac{1}{m} \sum_{k=-m}^{m} |z_m^\lambda(k)|^2.$$

Since

$$\|\tilde{h}_{\epsilon,\lambda}\|_{(1)}^2 = 2\pi \|\hat{h}_{\epsilon,\lambda}\|_{(\infty)}^2$$

we have

$$\|\tilde{h}_{\epsilon,\lambda}\|_{(1)}^2 \leqslant \overline{\lim}_{m\to\infty} \frac{1}{m} \sum_{k=-m}^{m} |z_m^\lambda(k)|^2.$$

By (3.12),

$$\|\tilde{h}_{\epsilon,\lambda}\|_{(1)} \leqslant \|g_{\epsilon,\mu}\|_{(1)} + \frac{\lambda - \mu}{3}.$$

Letting $\mu \to \lambda$, by the continuity of $h(\lambda)$ on (ϵ, ∞) in lemma 7, we have

$$\|\tilde{h}_{\epsilon,\lambda}\|_{(1)} \leqslant \|g_{\epsilon,\lambda}\|_{(1)}.$$

By (3.11), we have proved that

$$\|\tilde{h}_{\epsilon,\lambda}\|_{(1)} = \|g_{\epsilon,\lambda}\|_{(1)}.$$

Since $\tilde{h}_{\epsilon,\lambda} \in \mathcal{BT}_{\epsilon,\lambda}$, by lemma 2, we have

$$\tilde{h}_{\epsilon,\lambda}(s) = g_{\epsilon,\lambda}(s) \qquad \text{for } s \in [-1, 1], \text{ almost surely.}$$

This proves that Ψ_m^λ converges to $f_{\epsilon,\lambda}$ as $m \to \infty$. □

4. Band-limited signal spaces $\mathcal{BL}_\gamma$

The error estimate result in theorem 1 is for band-limited signals in the spaces $\mathcal{BL}_\gamma$. The conditions in (2.1)–(2.3) defining these spaces are rather abstract. In this section, we study their properties and simplifications. To do so, let us first review the prolate spheroidal wavefunctions (see [25, 29, 30]).

Let K be the following operator

$$(Kf)(t) = \int_{-T}^{T} \frac{\sin\Omega(t-\tau)}{\pi(t-\tau)} f(\tau)\,\mathrm{d}\tau \qquad f \in L^2[-T, T]. \tag{4.1}$$

It is clear that the operator K defined on $L^2[-T,T]$ is self-adjoint and compact. Let ϕ_k and λ_k, $k = 0, 1, 2, \ldots$, be the eigenfunctions and the corresponding eigenvalues of the operator K, respectively, such that ϕ_k, $k = 0, 1, 2, \ldots$, form an orthogonal basis for $L^2[-T,T]$ with

$$\int_{-T}^{T} \phi_j(t)\phi_k(t)\,\mathrm{d}t = \lambda_k \delta(j-k)$$

where $\delta(n) = 1$ when $n = 0$ and $\delta(n) = 0$ otherwise. Moreover, we have

$$1 > \lambda_0 > \lambda_1 > \cdots > 0 \qquad \text{and} \qquad \lambda_k \to 0 \text{ as } k \to \infty. \tag{4.2}$$

From (4.1),

$$\phi_k(t) = \frac{1}{\lambda_k}\int_{-T}^{T} \frac{\sin\Omega(t-\tau)}{\pi(t-\tau)}\phi_k(\tau)\,\mathrm{d}\tau \qquad t \in [-T,T] \qquad k = 0, 1, 2, \ldots. \tag{4.3}$$

Although the above eigenfunctions ϕ_k are only defined on the interval $[-T, T]$, they can be easily extended to the whole real line $\mathbb{R}$ by letting t take an arbitrary real value in formula (4.3). By doing so, it was proved in [29, 30] that the extended eigenfunctions ϕ_k for $t \in \mathbb{R}$ have the following orthonormality:

$$\int_{-\infty}^{\infty} \phi_j(t)\phi_k(t)\,\mathrm{d}t = \delta(j-k).$$

These extended eigenfunctions ϕ_k are called the *prolate spheroidal wavefunctions* in [29, 30]. It was also proved in [29, 30] that these prolate spheroidal wavefunctions ϕ_k, $k = 0, 1, 2, \ldots$, form an orthonormal basis for the Ω band-limited signal space $\mathcal{BL}$. Thus, any $f \in \mathcal{BL}$ can be expanded as

$$f(t) = \sum_{k=0}^{\infty} a_k\phi_k(t) \qquad t \in \mathbb{R} \tag{4.4}$$

where

$$a_k = \int_{-\infty}^{\infty} f(t)\phi_k(t)\,\mathrm{d}t = \frac{1}{\lambda_k}\int_{-T}^{T} f(t)\phi_k(t)\,\mathrm{d}t \tag{4.5}$$

and

$$\|f\|_{(\infty)}^2 = \sum_{k=0}^{\infty} a_k^2 \tag{4.6}$$

and

$$\|f\|_{(T)}^2 = \sum_{k=0}^{\infty} a_k^2\lambda_k. \tag{4.7}$$

We now have the following result.

Theorem 3. *Let f be an Ω band-limited function and have the expansion (4.4), (4.5). If*

$$\sum_{k=0}^{\infty} \frac{a_k^2}{\lambda_k^{1-2\gamma/3}} < \infty \qquad \textit{for some } \gamma,\ 0 \leqslant \gamma < \tfrac{1}{2}$$

then $f \in \mathcal{BL}_\gamma$.

Proof. For $A > 0$, let D_A be the truncation operator on $L^2(\mathbb{R})$: for $h \in L^2(\mathbb{R})$,

$$(D_A h)(t) = \begin{cases} h(t) & t \in [-A, A] \\ 0 & \text{otherwise.} \end{cases}$$

By (4.4) and (4.5), letting F denote the Fourier transform, we obtain

$$\begin{aligned} \hat{f}(\omega) = Ff(\omega) &= \sum_{k=0}^{\infty} a_k F\phi_k(\omega) \\ &= \sum_{k=0}^{\infty} a_k D_\Omega F D_T \phi_k(t)/\lambda_k \\ &= D_\Omega \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} F D_T \phi_k(t). \end{aligned}$$

Let

$$\hat{f}_n = \sum_{k=0}^{n} a_k F\phi_k = D_\Omega \sum_{k=0}^{n} \frac{a_k}{\lambda_k} F D_T \phi_k.$$

Then

$$f_n = \sum_{k=0}^{n} a_k \phi_k$$

and

$$\|f_n - f\|^2_{(\infty)} = \sum_{k=n+1}^{\infty} a_k^2$$

$$\|\hat{f}_n - \hat{f}\|^2_{(\Omega)} = 2\pi \sum_{k=n+1}^{\infty} a_k^2$$

and

$$\hat{f}_n = D_\Omega F\left(D_T \sum_{k=0}^{n} \frac{a_k}{\lambda_k} \phi_k \right).$$

Let

$$g_n = 2\pi D_T \sum_{k=0}^{n} \frac{a_k}{\lambda_k} \phi_k.$$

Then

$$\hat{f}_n = D_\Omega \frac{1}{2\pi} F g_n$$

and

$$\|g_n\|^2_{(T)} = \sum_{k=0}^{n} \frac{a_k^2}{\lambda_k^2} \|\phi_k\|^2_{(T)} = \sum_{k=0}^{n} \frac{a_k^2}{\lambda_k}.$$

Let

$$b_k^2 = \frac{a_k^2}{\lambda_k^{1-2\gamma/3}} \qquad k = 0, 1, 2, \ldots.$$

Then by the assumption

$$B \triangleq \sum_{k=0}^{\infty} b_k^2 < \infty$$

we have

$$\|g_n\|_{(T)}^2 = \sum_{k=0}^{n} b_k^2 \lambda_k^{-2\gamma/3}$$

and

$$\|\hat{f}_n - \hat{f}\|_{(\Omega)}^2 = 2\pi \sum_{k=n+1}^{\infty} b_k^2 \lambda_k^{1-2\gamma/3}.$$

By (4.2), for any $\delta > 0$, there exists N such that

$$\lambda_k^{1-2\gamma/3} \leqslant \delta \qquad \text{for } k \geqslant N+1$$

and

$$\lambda_k^{1-2\gamma/3} > \delta \qquad \text{for } k \leqslant N.$$

Then

$$\|\hat{f}_N - \hat{f}\|_{(\Omega)}^2 \leqslant 2\pi \sum_{k=N+1}^{\infty} b_k^2 \delta \leqslant 2\pi B\delta$$

and

$$\|g_N\|_{(T)}^2 \leqslant \sum_{k=0}^{N} b_k^2 \delta^{-\frac{2\gamma/3}{1-2\gamma/3}} \leqslant B\delta^{-\frac{2\gamma}{3-2\gamma}}.$$

For $0 \leqslant \gamma < \frac{1}{2}$, there exists a constant $C > 0$ such that

$$\delta^{\gamma} \|g_N\|_{(T)}^2 \leqslant C.$$

Let

$$\hat{f}_{\sqrt{2B\pi\delta}} = \frac{1}{2\pi} F g_N.$$

Then

$$\hat{f}_N = D_\Omega \hat{f}_{\sqrt{2B\pi\delta}}$$
$$\|\hat{f}_{\sqrt{2B\pi\delta}} - \hat{f}\|_{(\Omega)} \leqslant \sqrt{2B\pi\delta}$$

and

$$\begin{aligned}
\left(\sqrt{2B\pi\delta}\right)^{\gamma} \|\hat{f}_{\sqrt{2B\pi\delta}}\|_{(\infty)} &= \left(\sqrt{2B\pi\delta}\right)^{\gamma} \frac{1}{\sqrt{2\pi}} \|g_N\|_{(T)} \\
&\leqslant B^{\gamma/2} (2\pi)^{(\gamma-1)/2} \delta^{\gamma/2} \|g_N\|_{(T)} \\
&\leqslant B^{\gamma/2} C^{1/2} (2\pi)^{(\gamma-1)/2} \qquad \text{for } 0 \leqslant \gamma < \tfrac{1}{2}.
\end{aligned}$$

This proves that f satisfies (2.1)–(2.3). □

Before going to the next result, we recall a result on operator equations. Suppose that K is a compact linear operator from Hilbert space $\mathbb{H}_1$ to Hilbert space $\mathbb{H}_2$. Let $\theta_1^2 \geqslant \theta_2^2 \geqslant \cdots$ be the sequence of eigenvalues of the operator K^*K, and $v_1, v_2, \ldots$ be the associated orthonormal eigenfunction sequence. Let $\mu_n = \theta_n^{-1}$ and

$$u_n = \mu_n K v_n. \tag{4.8}$$

Then $\{u_n\}$ is an orthonormal sequence in $\mathbb{H}_2$ and

$$v_n = \mu_n K^* u_n. \tag{4.9}$$

We call the sequence $\{u_n, v_n; \mu_n\}$ *a singular system* for the operator K. Then, Picard's theorem can be stated as follows (for details, see, for example [10, 20]).

Proposition 2. *Let $K : \mathbb{H}_1 \to \mathbb{H}_2$ be a compact linear operator with singular system $\{u_n, v_n; \mu_n\}$. In order that the equation $Kz = g$ has a solution, it is necessary and sufficient that $g \in \mathrm{Ker}(K^*)^{\perp}(= \mathrm{Closure}\mathcal{R}(K))$ and*

$$\sum_{n=0}^{\infty} \mu_n^2 |\langle g, u_n\rangle|^2 < \infty$$

where $\langle\, ,\rangle$ is the inner product on $\mathbb{H}_2$.

We now have the following result.

Theorem 4. *Assume that f is Ω band limited and with expansion (4.4), (4.5). Then:*

(i) $f \in \mathcal{BL}_\gamma$ with $\gamma = 0$ if and only if its Fourier transform $\hat{f}(\omega)$ or $-\hat{f}(-\omega)$ for $\omega \in (-\Omega, \Omega)$ is a piece of T band-limited signal;

(ii) $f \in \mathcal{BL}_\gamma$ with $\gamma = 0$ if and only if

$$\sum_{k=0}^{\infty} \frac{a_k^2}{\lambda_k} < \infty.$$

Proof of (i). 'If part': If $-\hat{f}(-\omega)$ for $\omega \in (-\Omega, \Omega)$ is a piece of T band-limited signal, then there exists $g \in L^2[-T, T]$ such that

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-T}^{T} \mathrm{e}^{\mathrm{i}s\omega} g(s)\mathrm{d}s \qquad \omega \in (-\Omega, \Omega).$$

For any $\delta > 0$, let $g_\delta = g$. Then, $\hat{f}_\delta(\omega) = \hat{f}(\omega)$ for $\omega \in (-\Omega, \Omega)$. Let $C = \frac{1}{2\pi}\|g\|_{(T)}$. Then

$$\|\hat{f}_\delta - \hat{f}\|_{(\Omega)} = 0 \leqslant \delta$$

and

$$\|\hat{f}_\delta\|_{(\infty)} = \frac{1}{\sqrt{2\pi}}\|g_\delta\|_{(T)} = \frac{1}{\sqrt{2\pi}}\|g\|_{(T)} = C.$$

Thus $f \in \mathcal{BL}_\gamma$ for $\gamma = 0$.

'Only if part': If $f \in \mathcal{BL}_0$, then for every $\delta > 0$ there exists $g_\delta \in L^2[-T, T]$ such that

$$\|\hat{f}_\delta - \hat{f}\|_{(\Omega)} \leqslant \delta \qquad \text{and} \qquad \|g_\delta\|_{(T)} \leqslant 2\pi C$$

where C is a constant and

$$\hat{f}_\delta(\omega) = \frac{1}{2\pi}\int_{-T}^{T} g_\delta(s)\mathrm{e}^{\mathrm{i}s\omega}\mathrm{d}s.$$

Thus, the function family $\{\hat{f}_\delta\}$ is normal. In fact,

$$|\hat{f}_\delta(z)| \leqslant \frac{\sqrt{2T}}{2\pi} \mathrm{e}^{|z|T} \|g_\delta\|_{(T)} \leqslant C\sqrt{2T}\mathrm{e}^{T|z|} \qquad \text{for all } \delta > 0 \qquad z \in \mathbb{C}.$$

Therefore, for every sequence $\{\delta_n\}$ that tends to 0 when $n \to \infty$, there is a subsequence $\{\delta_{n_j}\}$ such that $\{\hat{f}_{\delta_{n_j}}\}$ converges to a T band-limited signal $\hat{h}$ uniformly on compact sets of $\mathbb{C}$. On the other hand,

$$\left| \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \mathrm{e}^{-\mathrm{i}t\omega} \hat{f}_{\delta_{n_j}}(\omega)\mathrm{d}\omega - f(t) \right| \leqslant \frac{\sqrt{2\pi}}{2\pi} \delta_{n_j}.$$

Letting $j \to \infty$, we obtain

$$\frac{1}{2\pi} \int_{-\Omega}^{\Omega} \mathrm{e}^{-\mathrm{i}t\omega} \hat{h}(\omega)\mathrm{d}\omega = f(t).$$

This proves $\hat{h}(\omega) = \hat{f}(\omega)$ for $\omega \in (-\Omega, \Omega)$, that is f is a piece of a T band-limited signal.

Proof of (ii). Let $\mathbb{H}_1 = L^2[-T, T]$ and $\mathbb{H}_2 = \mathcal{BL}_0$. The inner product on $\mathbb{H}_2$ is the usual $L^2(\mathbb{R})$ inner product. Let K be the integral operator given in (4.1). By part (i), $K(L^2[-T, T]) = \mathcal{BL}_0$. By theorem 3, all finite linear combinations of the eigenfunctions ϕ_k are in $\mathcal{BL}_0$. Thus, Closure$(\mathcal{BL}_0) = \mathcal{BL}$ and therefore, $\mathcal{BL} =$ Closure$(\mathcal{R}(K))$, where the closure is under the usual $L^2(\mathbb{R})$ norm. Also,

$$K^* f(t) = \int_{-\infty}^{\infty} \frac{\sin \Omega(s-t)}{\pi(s-t)} f(s)\, \mathrm{d}s \qquad \text{for } f \in \mathcal{BL}_0.$$

From (4.8) and (4.9),

$$u_n = \mu_n^2 K K^* u_n.$$

Hence, $\{\mu_n^2\}$ are eigenvalues of the operator KK^* and $\{u_n\}$ are the corresponding eigenfunctions. Since

$$K^*\phi_n(t) = \int_{-\infty}^{\infty} \frac{\sin \Omega(s-t)}{\pi(s-t)} \phi_n(s)\, \mathrm{d}s = \phi_n(t)$$

we have

$$KK^*\phi_n = K\phi_n = \lambda_n \phi_n.$$

Thus by the completeness of the sequence $\{\phi_n\}$ we have

$$\lambda_n = \mu_n^{-2} \qquad \text{and} \qquad \phi_n = u_n.$$

By proposition 2,

$$f \in \mathcal{BL}_0 \qquad \text{iff } \sum_{n=0}^{\infty} \lambda_n^{-1} |(f, \phi_n)|^2 < \infty \qquad \text{iff } \sum_{n=0}^{\infty} \frac{a_n^2}{\lambda_n} < \infty.$$

This proves (ii). □

Combining theorems 1, 3 and 4, we have the following corollaries.

Corollary 1. *For* $0 \leqslant \gamma < \frac{1}{2}$, *if*

$$f(t) = \sum_{k=0}^{\infty} a_k \phi_k(t) \qquad \textit{and} \qquad \sum_{k=0}^{\infty} \frac{a_k^2}{\lambda_k^{1-2\gamma/3}} < \infty$$

then,

$$|f_{\epsilon,2\epsilon}(t) - f(t)| \leqslant C\epsilon^{(1-2\gamma)/3} \qquad t \in \mathbb{R}.$$

Corollary 2. *Let f be Ω band limited. If its Fourier transform $\hat{f}(\omega)$ for $\omega \in (-\Omega, \Omega)$ is a piece of a T band-limited function, then*

$$|f_{\epsilon,2\epsilon}(t) - f(t)| \leqslant C\epsilon^{1/3} \qquad t \in \mathbb{R}.$$

5. Remarks

In [14, 17], approximations of Ω band-limited signals f are considered. These authors use finite data of f on $[-T, T]$ to recover the whole f on $[-T, T]$. The optimal algorithm in the worst case for the recovery has been found in [14, 17] as follows.

Let O_m be an information operator which is a mapping $O_m : \mathcal{BL} \to \mathbb{C}^m$,

$$O_m f = (f(t_1), f(t_2), \ldots, f(t_m)).$$

An algorithm Φ is a function-valued mapping on $O_m\mathcal{BL}$. The optimal algorithm using O_m in the worst case takes the form:

$$\Phi(O_m f) = \sum_{k=1}^{m} b_k \frac{\sin\Omega(\cdot - t_k)}{\cdot - t_k}$$

where the coefficients $b_1, b_2, \ldots, b_m$ are determined by the solution of the linear system

$$\sum_{k=1}^{m} b_k \frac{\sin\Omega(t_n - t_k)}{t_n - t_k} = f(t_n) \qquad n = 1, 2, \ldots, m.$$

We can see that this is similar to the discretization of the MMNS in (3.1)–(3.3).

As we have already stated, a band-limited signal is the restriction of an entire function to the real line. But it is more than this. The Paley–Wiener theorem (see [1]) gives a direct characterization of band-limited signals; namely, a signal in $L^2(\mathbb{R})$ is 2π band limited if and only if it is the restriction of an entire function and is of exponential order on the real line. This provides a powerful property for extrapolation of band-limited signals that distinguishes the problem within the realm of analytic continuation of analytic functions, and makes finer and stable recovery results possible.

There is considerable literature on uniform and nonuniform sampling theorems for the recovery of band limited and other classes of signals from a countable set of sample values (see [2, 3, 12, 23, 37]), the simplest and most celebrated version being the Shannon–Whittaker theorem, which asserts that a π band-limited signal can be reconstructed via the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin\pi(t-n)}{\pi(t-n)}.$$

Various error estimates (truncation, jitter, amplitude, and aliasing errors) are also known. The problem of signal extrapolation from an interval (which usually has a small length) is markedly different from the reconstruction of the signal f via a sampling expansion theorem (which utilizes values of f on an appropriate infinite sequence with no accumulation point).

As we have shown $\mathcal{BL}_0$ is the range of the Hilbert–Schmidt compact linear operator (4.1) on $L^2[-T, T]$. $\mathcal{BL}_0$ is nonclosed in $L^2[-T, T]$. Nashed and Wahba [21, 22] have shown that the range of a Hilbert–Schmidt compact operator K is a *reproducing kernel Hilbert space* (RKHS) $\mathcal{H}_Q$ with reproducing kernel

$$Q(t, s) = \int_{-T}^{T} K(t, u)K(s, u)\,\mathrm{d}u$$

where $K(t,u)$ is the Hilbert–Schmidt kernel. The inner product on $\mathcal{H}_Q$ is given by $\langle f_1, f_2\rangle_Q = \langle K^\dagger f_1, K^\dagger f_2\rangle$ for f_1, f_2 in $\mathcal{H}_Q$, where $K^\dagger$ is the Hilbert space (Moore–Penrose) generalized inverse. Equivalently,

$$\langle f_1, f_2\rangle_Q = \int_{-T}^{T} p_1(s)p_2(s)\,\mathrm{d}s$$

where p_i is the element of the minimal norm which satisfies $Kp = f_i$, corresponding to f_i in $\mathcal{BL}_0$ for $i = 1, 2$. We recall that a Hilbert space $\mathbb{H}$ of functions f on an interval $\mathbb{J}$ is said to be a RKHS if all the evaluation functionals $E_t(f) = f(t)$, $f \in \mathbb{H}$, for each fixed $t \in \mathbb{J}$, are continuous. Then by the Riesz's representation theorem, for each $t \in \mathbb{J}$, there exists a unique element, call it Q_t, in $\mathbb{H}$ such that $f(t) = \langle f, Q_t\rangle$, $f \in \mathbb{H}$, where $\langle\,,\rangle$ is the inner product on $\mathbb{H}$. Let $Q(t,s) = \langle Q_s, Q_t\rangle$ for s, t in $\mathbb{J}$; this is the reproducing kernel (RK) of $\mathbb{H}$, and the space $\mathbb{H}$ with RK $Q(t,s)$ is denoted by $\mathbb{H}_Q$. The space $L^2(\mathbb{J})$ is not a RKHS.

The Paley–Wiener space $\mathcal{BL}$ of band-limited signals with band $[-\pi,\pi]$ is a RKHS with RK

$$Q(t,s) = \frac{\sin\pi(t-s)}{\pi(t-s)}.$$

In [23] it is shown that there is a strong affinity between RK Hilbert spaces and sampling theorems, and general sampling theorems were established for signals belonging to a RKHS which is also a closed subspace of the Sobolev space $\mathbb{H}^{-1}$. The preceding remarks about $\mathcal{BL}_0$ and the other related spaces being RKHS may suggest that a broader framework within which the type of extrapolation results derived in this paper may also hold.

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