On Generalized-Marginal Time-Frequency Distributions

Xiang-Gen Xia, Yuri Owechko, Bernard H. Soffer, and Roy M. Matic

Abstract— **In this correspondence, we introduce a family of time-frequency (TF) distributions with generalized marginals, i.e., beyond the time-domain and the frequency-domain marginals, in the sense that the projections of a TF distribution along one or more angles are equal to the magnitude squared of the fractional Fourier transforms of the signal. We present a necessary and sufficient condition for a TF distribution in Cohen's class to satisfy generalized marginals. We then modify the existing well-known TF distributions in Cohen's class, such as Choi–Williams and Page distributions, so that the modified ones have generalized marginals. Numerical examples are presented to show that the proposed TF distributions have the advantages of both Wigner–Ville and other quadratic TF distributions, which only have the conventional marginals. Moreover, they also indicate that the generalized-marginal TF distributions with proper marginals are more robust than the Wigner–Ville and the Choi–Williams distributions when signals contain additive noises.**

I. Introduction

Joint time-frequency (TF) representations have recently attracted considerable attention and have been applied in nonstationary signal processing, such as speech signal analysis. Several kinds of joint TF representations exist including linear transforms such as short time Fourier transforms (STFT) [9]–[11], [18], time-scale (wavelet) representations [12], [18], and bilinear transforms, such as Cohen's class [8]–[10] including Wigner-Ville, spectrogram [9]–[11], [18], Choi–Williams [13], and Zhao–Atlas–Marks [14] distributions, as well as positive distributions [15], adaptive TF distributions [16], distribution series [17], and the affine, hyperbolic, and power classes of TF representations [38]–[39]. TF distributions in Cohen's class are distinguished by different kernels and properties. For TF distributions, the marginal properties are important. That is, the integrals of a TF distribution along the time t and the frequency ω are the powers of the signal in the frequency and the time domains, respectively. Satisfaction of the time and the frequency marginals in Cohen's class is equivalent to the property of the kernels $\phi(\theta, \tau)$: $\phi(0, \tau) = \phi(\theta, 0) = 1$ for all real θ and τ; see, for example, [10].

It is known that when signals are chirp signals, the TF distributions should be concentrated on lines in the TF plane. The question is the following. If we have some prior information about a signal, can we take advantage of it in the design of a TF distribution, or can we put some requirements on a TF distribution along these lines? If so, how? For dechirping, the Radon–Wigner distribution was proposed in [1]–[3], where the Radon transform was used in the Wigner–Ville distribution domain. In [4]–[7] and [38] and [39], joint distributions for arbitrary variables and unitary transformations of the time and the frequency variables were studied. In [38] and [39], generalized-marginal TF representations were mentioned in a different way from this paper.

In this correspondence, we want to study TF distributions with generalized marginals beyond the usual time and frequency marginals. One might ask what the other marginals are. To answer this question, we recall the fractional Fourier transform $\boldsymbol{F}_\alpha$ with angle α studied in [19]–[34], and [41], which is a rotation of the time-frequency plane. When the angle α is $\pi/2$, $\boldsymbol{F}_\alpha$ is equal to the Fourier transform $\boldsymbol{F}$, i.e., the FRFT with angle $\pi/2$ of a signal is its Fourier transform. When the angle α is 0, $\boldsymbol{F}_\alpha$ is the identity transform, i.e., the FRFT with angle 0 of a signal is the signal itself. We now define the following generalized-marginal property. Let $P(t, \omega)$ be a TF distribution of a signal $s(t)$. Let L_α denote a straight line on the time-frequency plane through the origin with angle α (see Fig. 1). Let $L_\alpha(u)$ denote one member of the family of all parallel lines of L_α parameterized by a real number u (see Fig. 1). We call $P(t, \omega)$ a *generalized-marginal time-frequency distribution* if the line integrals of $P(t, \omega)$ along the lines $L_{\alpha_k}(u)$ for $k = 1, 2, \cdots, N$ are the powers of the FRFT with angles $\alpha_k + \pi/2, k = 1, 2, \cdots, N$, of the signal s, respectively. In other words,

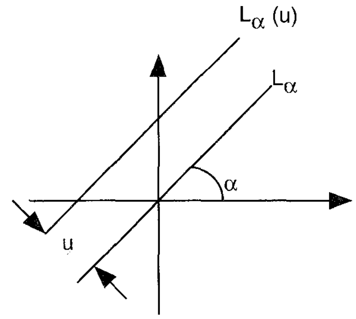

Fig. 1. Line L_α.

$$\int_{L_{\alpha_k}(u)} P(t, \omega)\, dx = |(\boldsymbol{F}_{\alpha_k+\pi/2}s)(u)|^2, k = 1, 2, \cdots, N \quad (1)$$

or simply,

$$\int_{L_{\alpha_k}} P(t, \omega)\, dx = |\boldsymbol{F}_{\alpha_k+\pi/2}s|^2, \qquad k = 1, 2, \cdots, N.$$

It is clear that when $\alpha_1 = 0, \alpha_2 = \pi/2$, and $N = 2$, the above generalized marginals are the conventional marginals. In addition, the angles α_k may be chosen to be close to the angles of chirp signals in the TF plane.

In this correspondence, we will study TF distributions with kernels $\phi(\theta, \tau)$ in Cohen's class, which are generalized-marginal TF distributions (1). We show that a TF distribution with kernel $\phi(\theta, \tau)$ in Cohen's class is a generalized-marginal one (1) if and only if its kernel $\phi(\theta, \tau)$ is equal to 1 on the lines that are perpendicular to $L_{\alpha_k}, k = 1, 2, \cdots, N$, and pass through the origin. This implies that the Wigner–Ville distribution satisfies all the generalized marginals, and it is the only one in Cohen's class with this property. We then modify existing TF distributions with the conventional marginals in Cohen's class so that the modified ones are generalized marginal. We then present some numerical examples of generalized-marginal TF distributions, which show that they combine the advantages of the Wigner–Ville and other bilinear TF distributions. Our numerical examples also show that the generalized-marginal TF distributions with proper marginals are more robust and cleaner than the Wigner–Ville and Choi–Williams distributions when signals are disturbed by additive noise.

This correspondence is organized as follows. In Section II, we study generalized-marginal TF distributions in Cohen's class. In Section III, we present numerical examples.

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The authors are with Hughes Research Laboratories, 3011 Malibu Canyon Rd., Malibu, CA 90265 USA.

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II. Bilinear Generalized-Marginal Time-Frequency Distributions

In this section, we first briefly review fractional Fourier transform and then study generalized-marginal time-frequency distributions.

A. Fractional Fourier Transform

Basically, the FRFT is a rotation of the time-frequency plane. For any real α, the fractional Fourier transform $\boldsymbol{F}_\alpha$ with angle α is defined by

$$(\boldsymbol{F}_\alpha s)(u) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} \cdot e^{j(u^2/2)\cot\alpha} \int_{-\infty}^{\infty} s(t) e^{j(t^2/2)\cot\alpha} \\ \cdot e^{-jut\csc\alpha}\, dt, & \text{if } \alpha \text{ is not a multiple of } \pi, \\ s(u), & \text{if } \alpha \text{ is a multiple of } 2\pi, \\ s(-u), & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi. \end{cases} \quad (2)$$

Additionally, one can see that $\boldsymbol{F}_{2n\pi}$ is the identity transformation $(\boldsymbol{F}_{(2n+1)\pi} s)(t) = s(-t)$, and $\boldsymbol{F}_{\pi/2}$ is the traditional Fourier transform. Moreover, the following rotation property holds: $\boldsymbol{F}_{\alpha+\beta} = \boldsymbol{F}_\alpha \boldsymbol{F}_\beta$; see [19] and [20]. For more properties, see, for example, [19]–[34], [41].

With the FRFT, it was proved in [21]–[23], [28], [29] that a rotation of a Wigner–Ville distribution is still a Wigner–Ville distribution as follows.

Let $P_W(t,\omega)$ denote the Wigner–Ville distribution of a signal $s(t)$, i.e.,

$$P_W(t,\omega) = \int s\Big(t+\frac{\tau}{2}\Big) s^*\Big(t-\frac{\tau}{2}\Big) e^{-j\omega\tau}\, d\tau. \quad (3)$$

Let α be an angle and $(\tilde{t},\tilde{\omega})$ be a rotation of (t,ω) with angle α:

$$\tilde{t} = t\cos\alpha + \omega\sin\alpha,$$
$$\tilde{\omega} = -t\sin\alpha + \omega\cos\alpha \quad (4)$$

and

$$\tilde{P}_W(\tilde{t},\tilde{\omega}) = P_W(\tilde{t}\cos\alpha - \tilde{\omega}\sin\alpha, \tilde{t}\sin\alpha + \tilde{\omega}\cos\alpha).$$

Then, see, for example, [22]

$$\tilde{P}_W(\tilde{t},\tilde{\omega}) = \int (\boldsymbol{F}_\alpha s)\Big(\tilde{t}+\frac{\tau}{2}\Big)(\boldsymbol{F}_\alpha s)^*\Big(\tilde{t}-\frac{\tau}{2}\Big) e^{-j\tilde{\omega}\tau}\, d\tau. \quad (5)$$

Equation (5) tells us that the rotation $\tilde{P}_W(\tilde{t},\tilde{\omega})$ of the Wigner–Ville distribution $P_W(t,\omega)$ of a signal s is the Wigner–Ville distribution of the signal $\boldsymbol{F}_\alpha s$. It was also proved in [24] that a rotation of a Radon–Wigner distribution is also a Radon–Wigner distribution by using the FRFT technique. Rotations of TF distributions in Cohen's class were studied in [32]. It is known that a Wigner–Ville distribution satisfies the conventional marginal properties for $\alpha = \pi/2, 0$. Thus

$$\int \tilde{P}_W(\tilde{t},\tilde{\omega})\, d\tilde{\omega} = |(\boldsymbol{F}_\alpha s)(\tilde{t})|^2 \quad (6)$$

and

$$\int \tilde{P}_W(\tilde{t},\tilde{\omega})\, d\tilde{t} = |\boldsymbol{F}(\boldsymbol{F}_\alpha s)(\tilde{\omega})|^2 = |(\boldsymbol{F}_{\alpha+\pi/2} s)(\tilde{\omega})|^2. \quad (7)$$

B. A Necessary and Sufficient Condition

We now derive a necessary and sufficient condition for a TF distribution in Cohen's class to have the generalized-marginal property.

A TF distribution for a signal $s(t)$ in Cohen's class is defined by

$$P(t,\omega) = \frac{1}{4\pi^2} \iint A(\theta,\tau) e^{-j\theta t - j\tau\omega}\, d\theta\, d\tau$$

where $A(\theta,\tau)$ is the generalized ambiguity function of the signal $s(t)$ with a kernel $\phi(\theta,\tau)$:

$$A(\theta,\tau) = \phi(\theta,\tau) \int s\Big(u+\frac{\tau}{2}\Big) s^*\Big(u-\frac{\tau}{2}\Big) e^{j\theta u}\, du.$$

The TF distribution $P(t,\omega)$ can be also written as

$$P(t,\omega) = \iiint e^{-j\theta t - j\tau\omega + j\theta u} \phi(\theta,\tau) \cdot s\Big(u+\frac{\tau}{2}\Big) s^*\Big(u-\frac{\tau}{2}\Big)\, d\theta\, d\tau\, du. \quad (8)$$

Then, $P(t,\omega)$ is a generalized-marginal TF distribution if and only if the following holds. Let

$$\tilde{t} = t\cos\alpha + \omega\sin\alpha, \quad \tilde{\omega} = -t\sin\alpha + \omega\cos\alpha, \quad \text{or} \quad t = \tilde{t}\cos\alpha - \tilde{\omega}\sin\alpha, \quad \omega = \tilde{t}\sin\alpha + \tilde{\omega}\cos\alpha.$$

Then, the condition (1) is equivalent to

$$\int P(\tilde{t}\cos\alpha_k - \tilde{\omega}\sin\alpha_k, \tilde{t}\sin\alpha_k + \tilde{\omega}\cos\alpha_k)\, d\tilde{t} = |(\boldsymbol{F}_{\alpha_k+\pi/2} s)(\tilde{w})|^2, \qquad k = 1, 2, \cdots, N. \quad (9)$$

In other words, a TF distribution $P(t,\omega)$ is generalized marginal if and only if it satisfies (9). We now focus on the TF distributions $P(t,\omega)$ in (8) and one angle α in (9). Let us see what the left-hand side of (9) with angle α for $P(t,\omega)$ in (8) is.

$$\int P(\tilde{t}\cos\alpha - \tilde{\omega}\sin\alpha, \tilde{t}\sin\alpha + \tilde{\omega}\cos\alpha)\, d\tilde{t}$$
$$= \frac{1}{\cos\alpha} \iint e^{-j\tilde{\omega}\tau(\sin\alpha\tan\alpha + \cos\alpha) - ju\tau\tan\alpha} \cdot \phi(-\tau\tan\alpha, \tau) s\Big(u+\frac{\tau}{2}\Big) s^*\Big(u-\frac{\tau}{2}\Big)\, d\tau\, du$$
$$= \int e^{-j\tilde{\omega}\tau} \phi(-\tau\sin\alpha, \tau\cos\alpha) \cdot A_s(-\tau\sin\alpha, \tau\cos\alpha)\, d\tau$$

where A_s is the ambiguity function of s. It was proved (see, for example, [37]) that

$$A_s(-\tau\sin\alpha, \tau\cos\alpha) = A_{\boldsymbol{F}_{\alpha+\pi/2} s}(\tau, 0).$$

Therefore,

$$\int P(\tilde{t}\cos\alpha - \tilde{\omega}\sin\alpha, \tilde{t}\sin\alpha + \tilde{\omega}\cos\alpha)\, d\tilde{t} = \iint e^{j(u-\tilde{\omega})\tau} \phi(-\tau\sin\alpha, \tau\cos\alpha) \cdot |\boldsymbol{F}_{\alpha+\pi/2} s(u)|^2\, du\, d\tau.$$

Therefore, the generalized-marginal property

$$\int P(\tilde{t}\cos\alpha - \tilde{\omega}\sin\alpha, \tilde{t}\sin\alpha + \tilde{\omega}\cos\alpha)\, d\tilde{t} = |\boldsymbol{F}_{\alpha+\pi/2} s(\tilde{\omega})|^2$$

holds if and only if

$$\int e^{j(u-\tilde{\omega})\tau} \phi(-\tau\sin\alpha, \tau\cos\alpha)\, d\tau = \delta(u-\tilde{\omega})$$

i.e., $\phi(-\tau\sin\alpha, \tau\cos\alpha) = 1$.

Although the above discussion is for one angle only, it is straightforward to generalize it to several angles α_k for $k = 1, 2, \cdots, N$. Therefore, we have proved the following main result.

Theorem 1: A time-frequency distribution $P(t,\omega)$ in (8) in Cohen's class with a kernel $\phi(\theta,\tau)$ is a generalized-marginal time-frequency distribution with angles $\alpha_k, k = 1, 2, \cdots, N$, as in (9) if and only if

$$\phi(-\tau\sin\alpha_k, \tau\cos\alpha_k) = 1, \textit{for all real } \tau, \textit{ and } k = 1, 2, \cdots, N. \quad (10)$$

In other words, ϕ is 1 on the lines perpendicular to the lines $L_{\alpha_k}, k = 1, 2, \cdots, N$, passing through the origin.

TABLE I
GENERALIZED-MARGINAL KERNELS

Name	Kernel $\phi(\theta,\tau)$	Generalized-Marginal $\tilde{\phi}(\theta,\tau)$ with angles α_k, $1 \le k \le N$.
Margenau-Hill	$\cos(0.5\theta\tau)$	$\cos[0.5\prod_{k=1}^{N}(\theta\cos\alpha_k + \tau\sin\alpha_k)]$
Kirkwood-Rihaczek	$e^{0.5j\theta\tau}$	$e^{0.5j\prod_{k=1}^{N}(\theta\cos\alpha_k+\tau\sin\alpha_k)}$
sinc	$\frac{\sin a\theta\tau}{a\theta\tau}$	$\frac{\sin[a\prod_{k=1}^{N}(\theta\cos\alpha_k+\tau\sin\alpha_k)]}{a\prod_{k=1}^{N}(\theta\cos\alpha_k+\tau\sin\alpha_k)}$
Page	$e^{0.5\theta\|\tau\|}$	$e^{0.5j\prod_{k_1=1}^{N_1}\|\theta\cos\alpha_{k_1}+\tau\sin\alpha_{k_1}\|\prod_{k_2=1}^{N_2}(\theta\cos\alpha_{k_2}+\tau\sin\alpha_{k_2})}$
Choi-Williams	$e^{-\frac{\theta^2\tau^2}{\sigma}}$	$e^{-\frac{1}{\sigma}\prod_{k=1}^{N}(\theta\cos\alpha_k+\tau\sin\alpha_k)^2}$

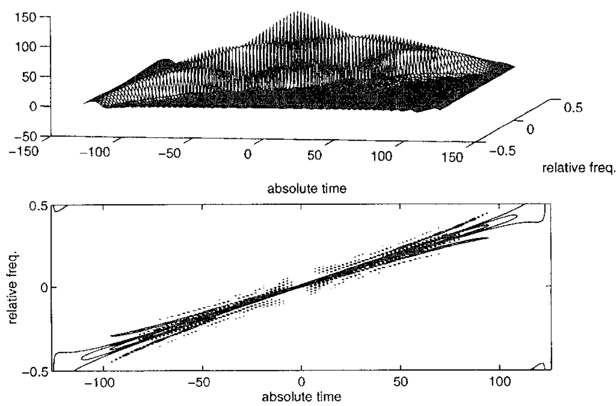

Fig. 2. Wigner–Ville distribution for the test signal $s_1(t)$.

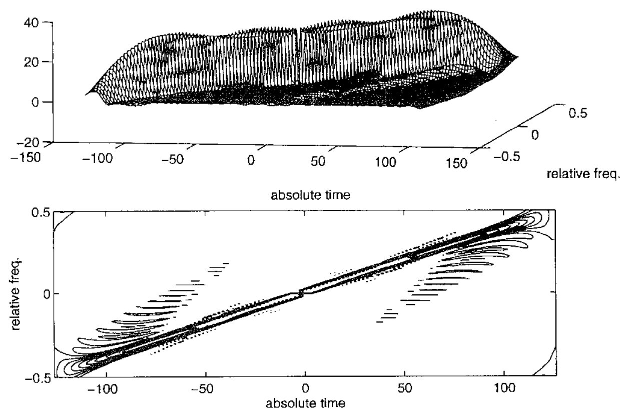

Fig. 3. Choi–Williams distribution for the test signal $s_1(t)$.

With this result, one can easily modify the well-known kernels so that the corresponding modified TF distributions are generalized marginals. We now list them in Table I. The motivation for this modification is similar to the one for existing distribution kernels [9]–[11], [13], [14], [18], [40].

Notice that there are some overlaps between the generalized-marginal Page distributions in Table I and the multiform, tiltable exponential distributions (MTED) proposed in [40].

Theorem 1 also tells us that the Wigner–Ville distribution satisfies all marginal properties for all angles because $\phi(\theta,\tau) = 1$ for all real θ and τ. There is, however, a tradeoff between the number of generalized marginals one wants to impose and the freedom of choosing a kernel $\phi(\theta,\tau)$ in Cohen's class. As more generalized-marginals are required, there is less freedom in choosing kernels.

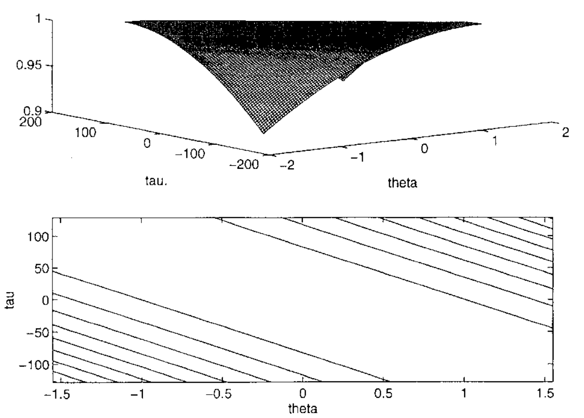

Fig. 4. Generalized marginal kernel $\phi(\theta,\tau)$ modified from the Choi–Williams kernel with only one angle $\tan\alpha_1 = 0.11^2$.

III. NUMERICAL EXAMPLES

In this section, we want to show some numerical examples for the existing TF distributions, such as the Wigner–Ville and Choi–Williams distributions with $\sigma = 128^2$ and their corresponding generalized-marginal TF distributions in Table I.

We test three signals. The first one is $s_1(n) = e^{0.5j(0.11n)^2}$ for $-64 \le n \le 64$ and 0 otherwise. Its Wigner-Ville and Choi-Williams distributions are shown in Figs. 2 and 3. Two generalized-marginal TF distributions modified from the Choi-Williams distribution are presented with two sets of angles. The first set of angles has only α_1 with $\tan\alpha_1 = 0.11^2$. Its corresponding kernel $\phi(\theta,\tau)$ is shown in Fig. 4. The modified Choi-Williams distribution with this kernel is shown in Fig. 5. The second set of angles has the above α_1 and the two conventional angles $\alpha_2 = 0$ and $\alpha_3 = \pi/2$. The corresponding kernel $\phi(\theta,\tau)$ is shown in Fig. 6 and the modified Choi-Williams distribution is shown in Fig. 7. One can see that the modified Choi-William distributions with generalized-marginals are as good as the Wigner-Ville distribution while they are cleaner than the Choi-Williams distribution.

The reason why the angle is choosen so small is because of the nature of chirp signals and the sampling in the calculations. If the sampling is not dense enough, the calculation will not be accurate due to the quadratic term t^2 in chirp signals unless the instantaneous frequencies, or equivalently the angles, are small enough. Without loss of generality, in the following numerical examples, we only use small angles with the limited sampling rate for simplicity in calculations.

The second test signal is $s_2(n) = s_1(n) + \eta(n)$, where η is Gaussian noise with maximum magnitude 0.2. Figs. 8–9 show its Wigner–Ville and Choi–Williams distributions. Figs. 10 and 11 show its modified Choi–Williams distributions with a single angle α_1 as

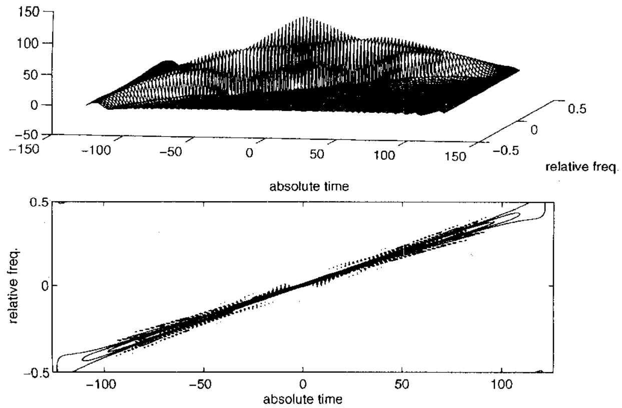

Fig. 5. Generalized marginal TF distribution with the kernel shown in Fig. 4 for the test signal s_1.

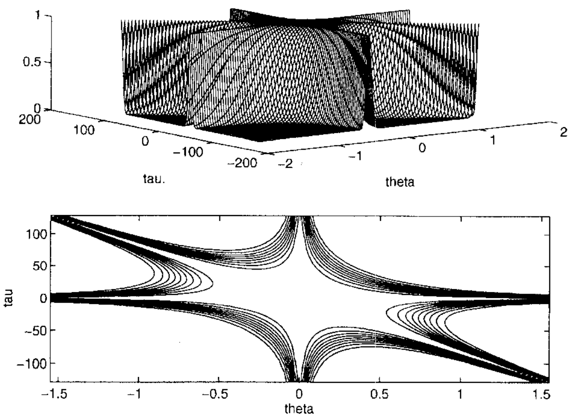

Fig. 6. Generalized marginal kernel $\phi(\theta, \tau)$ modified from the Choi–Williams kernel with angles $\tan \alpha_1 = 0.11^2, \alpha_2 = 0, \alpha_3 = \pi/2$.

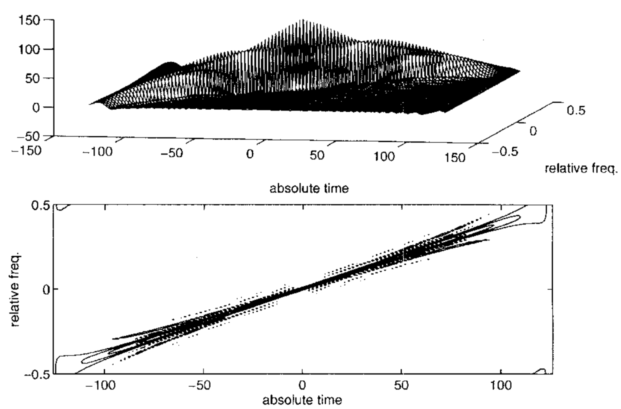

Fig. 7. Generalized marginal TF distribution with the kernel shown in Fig. 6 for the test signal s_1.

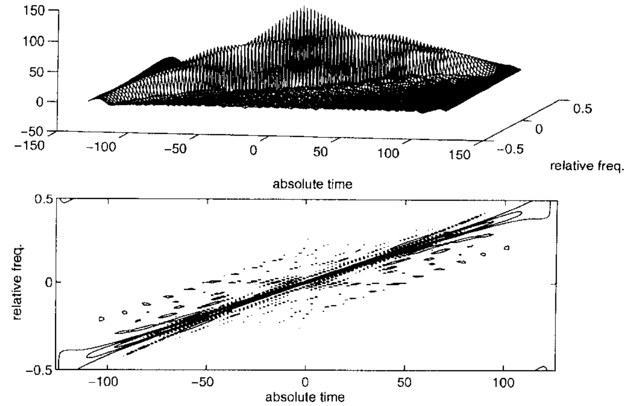

Fig. 8. Wigner–Ville distribution for the test signal $s_2(t)$ of the chirp signal s_1 with additive Gaussian noise.

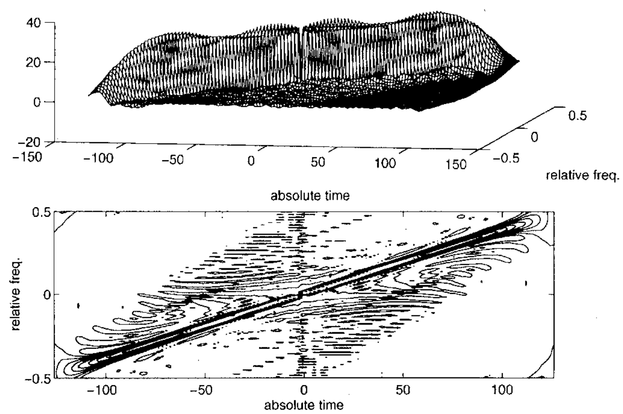

Fig. 9. Choi–Williams distribution for the test signal $s_2(t)$ of the chirp signal s_1 with additive Gaussian noise.

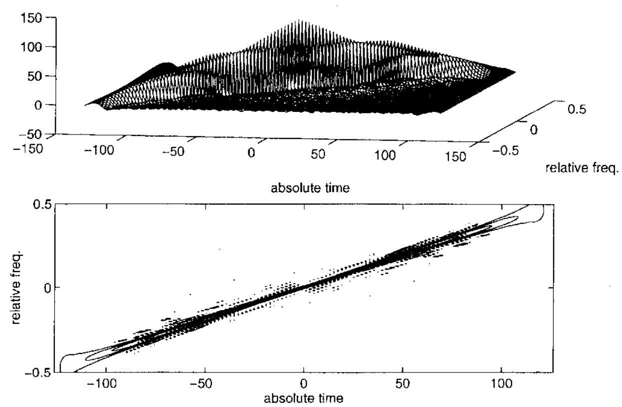

Fig. 10. Generalized marginal TF distribution with angle $\alpha_1 = \arctan 0.11^2$ for the test signal s_2.

$\tan \alpha_1 = 0.11^2$, and 0.09^2. We can see that the contours are much cleaner than the ones for the Wigner–Ville and the Choi–Williams distributions and are quite stable in terms of small change of the angle α_1.

IV. Conclusions

In this paper, we have generalized the conventional marginal properties to generalized-marginal properties by introducing generalized-marginal time-frequency (TF) distributions. We have characterized all generalized-marginal TF distributions in Cohen's class and modified the existing well-known TF distributions so that the modified ones have generalized-marginal properties. Numerical examples have been presented to show the performance of generalized-marginal TF distributions. They indicate that the TF distributions may have better performance by choosing proper generalized marginals and are stable in terms of small changes of angles in the generalized marginality. Numerical examples for noisy signals have been also given. They show that the generalized-marginal TF distributions

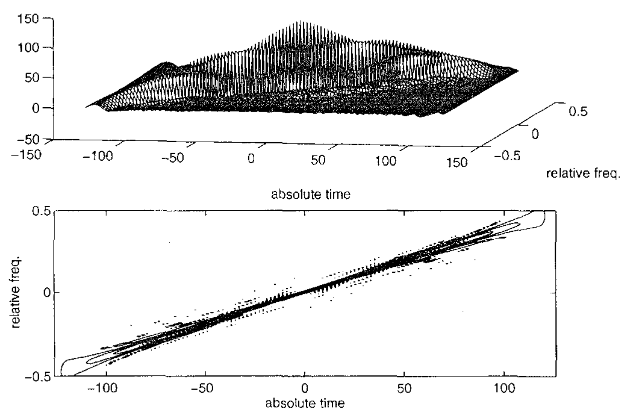

Fig. 11. Generalized marginal TF distribution with angle $\alpha_1 = \arctan 0.09^2$ for the test signal s_2.

with proper marginals are much cleaner than the Wigner–Ville and the Choi–Williams distributions. We believe that the proposed generalized marginal TF distributions have potential applications in multicomponent chirp signal detections. Future work on this direction would be interesting.

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