On Optimal Ambiguity Resistant Precoders in ISI/Multipath Cancellation

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***Abstract*—Ambiguity resistant (AR) precoding has recently been proposed in intersymbol interference (ISI) and multipath cancellations, where the ISI/multipath channel may have frequency-selective fading characteristics and its knowledge is not necessarily known. With the AR precoding, no diversity is necessary at the receiver. In the precoding, the AR property for a precoder plays an important rule. In this paper, more families and properties of AR precoders are presented and characterized. In particular, all systematic AR precoders are characterized. More importantly, we introduce the concepts of precoder *distance* and *optimal* precoders, and characterize and construct all optimal systematic AR precoders, when additive channel random noise is concerned. A necessary and sufficient condition for an AR precoder to be optimal is given, which is easy to check. With the optimal precoders, numerical simulations are presented to show the improved performance over the known AR precoders in ISI cancellation applications.**

I. Introduction

INTERSYMBOL interference (ISI) and multipath fading are important problems in digital communications. Precoding is one of the techniques for the ISI/multipath cancellation. The conventional precoding techniques [1]–[12], such as Tomlinson–Harashima (TH) precoding [1]–[2] and trellis precoding [3]–[5], may not perform well when the channel signal-to-noise ratio (SNR) is not high, and other ISI cancellation techniques [13]–[19], such as decision feedback equalizers, usually suffer from the spectrum-null characteristics in frequency-selective fading channels. Meanwhile, the conventional precoding methods require the knowledge of the ISI channel at the transmitter, i.e., a feedback channel is needed. Recently, a new precoding technique has been introduced in [20]–[23]. Unlike conventional precoding, the new precoding expands the bandwidth in a minimum amount as an expense. The advantages of the new precoding are the following: when there is no other noise but the ISI, it provides an ideal linear finite-impulse response (FIR) equalizer at the receiver, whether or not the ISI channel has spectrum-null; it is channel independent, i.e., the feedback channel is not necessary. It is linear (no modulo operation is needed); the transmitter or receiver does not have to know the ISI channel for the equalization, i.e., blind equalization is possible.

For the blind equalization with the new precoding technique, no diversity at the receiver is needed for a single receiver system, and a reduced sampling rate over the baud rate can be achieved in an antenna array receiver system, which are not possible for the existing blind equalization techniques (see for example [24]–[41]), without using precoding. For this purpose ambiguity resistant (AR) precoders have been introduced in [22]–[23] for combating the ambiguity induced by the ISI channel. Besides the AR precoder concept, some properties and a family of AR precoders are presented in [22]–[23].

In this paper, we further study AR precoders with more properties and more AR precoder families. All systematic AR precoders are characterized. More importantly, the concept of the *optimal* precoders is introduced when additive channel random noise is concerned. The optimality is based on the following criterion: the output symbols after the precoding should be as far away from each other as possible in the mean-square sense. This criterion is similar to the one in the modulation symbol design in communication systems to resist random errors. Given a precoder $\boldsymbol{G}(z)$, a polynomial matrix of the delay variable z^{-1}, its *distance* is introduced by using the coefficients of its coefficient matrices. It is proven that the distance is proportional to the mean distance of the ISI channel output symbols, which controls the performance in resisting additive channel random noise. We then characterize all optimal systematic AR precoders. A necessary and sufficient condition for an AR precoder to be optimal is given, which is easy to check. Numerical examples are presented to illustrate the theory.

This paper is organized as follows. In Section II, we briefly recall the concept of AR precoders and their applications in the ISI cancellation. In Section III, we present more properties and families of AR precoders, in particular AR systematic precoders. We introduce equivalent classes based on the AR property between AR precoders. In Section IV, we introduce the concepts of precoder distance and optimal AR precoders, study some properties, and characterize all optimal systematic AR precoders. In Section V, we present some simulation results. Finally, in Section VI, we conclude this paper.

II. AR Precoders via ISI Cancellation

A precoded single receiver system and undersampled antenna array receiver system are shown in Figs. 1 and 2, respectively, where $\tilde{\boldsymbol{G}}(z)$ in Fig. 1 and $\boldsymbol{G}(z)$ in Fig. 2 are precoders, $H(z)$, $H_1(z), \ldots, H_M(z)$ are the ISI channel transfer functions, and

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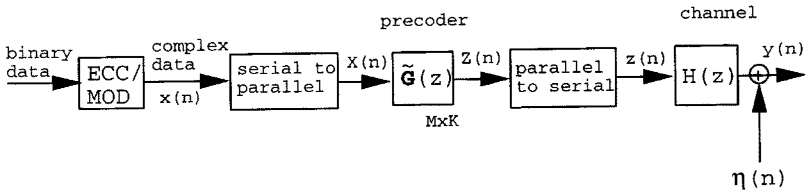

Fig. 1. Single antenna receiver with baud sampling rate.

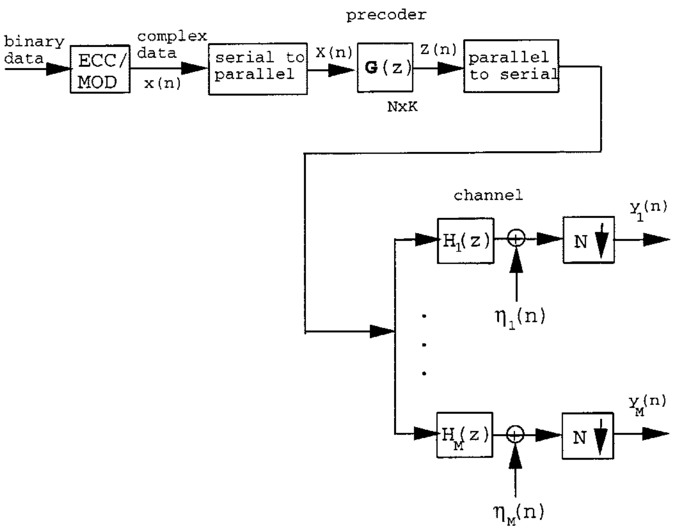

Fig. 2. An undersampled antenna array receiver system.

all of them are either polynomial matrices or polynomials of the delay variable z^{-1}. In what follows, boldface captial letters denote polynomial matrices.

Since the two systems in Figs. 1 and 2 can be converted to two multi-input multi-output (MIMO) systems, the existing MIMO system identification techniques (see for example [35]–[41]) can be used. However, based on these results on MIMO system identification, one can at most identify an MIMO system to a constant matrix ambiguity. In order to further resist the constant matrix ambiguity induced from an MIMO system identification algorithm, ambiguity resistant precoding has been introduced in [22]. A precoder $\boldsymbol{G}(z)$ of size $N \times K$ is considered *AR* if:

1) $\boldsymbol{G}(z)$ is irreducible, i.e., matrix $\boldsymbol{G}(z)$ has full rank for all complex values z including $z = \infty$;
2) the following equation for $K \times K$ polynomial matrix $\boldsymbol{V}(z)$ has only trivial solution $\boldsymbol{V}(z) = \alpha I_K$ for a nonzero constant α:

$$E\boldsymbol{G}(z) = \boldsymbol{G}(z)\boldsymbol{V}(z) \tag{2.1}$$

where E is an $N \times N$ nonzero constant matrix and I_K is the $K \times K$ identity matrix.

For an AR precoder $\boldsymbol{G}(z)$, the following lemma is proven in [22].

Lemma 1: If an $N \times K$, $K > 1$, $\boldsymbol{G}(z)$ is AR, then:

1) *there exists no full-rank constant matrix E and invertible $K \times K$ polynomial matrix $\boldsymbol{V}(z)$, such that the first column in matrix $E\boldsymbol{G}(z)\boldsymbol{V}(z)$ is $(1, 0, 0, \ldots, 0)^T$;*
2) $N > K$.

The above necessary condition 2) for an AR precoder means that the precoding has to expand each K samples into N samples. This is intuitively clear that certain redundancy is needed to resist errors. In a bandlimited channel, the minimum bandwidth expansion is desired. This implies that the optimal parameter K should be $K = N - 1$ given N in an AR precoder.

Let $\tilde{\boldsymbol{G}}(z)$ in Fig. 1 take the following form:

$$\tilde{\boldsymbol{G}}(z) = \begin{bmatrix} I_N \\ 0_{(M-N)\times N} \end{bmatrix} \boldsymbol{G}(z)$$

where $M > N$ and $\boldsymbol{G}(z)$ are an $N \times K$ polynomial matrix. It has been proven in [22] and [23] that, if the precoders in Figs. 1 and 2 take the above forms and $\boldsymbol{G}(z)$ are AR, then under certain conditions on the channel order, the input signals in the systems in Figs. 1 and 2 can be blindly identified from the output signals, where the ISI channels $H(z), H_1(z), \ldots, H_M(z)$ may have spectrum-null. In [22]–[23], some closed-form blind identification algorithms have also been obtained.

In [22], it is proven that the following precoders $\boldsymbol{G}(z)$ of sizes $N \times (N-1)$ are AR:

$$\boldsymbol{G}(z) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ z^{-r-1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & z^{-r-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-r-1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & z^{-r-1} \end{bmatrix}_{N\times(N-1)} \tag{2.2}$$

for an integer $r \geq 0$. From known AR precoders, the following lemma is straightforward, which can be used to build more AR precoders.

Lemma 2: If an $N \times K$ polynomial matrix $\boldsymbol{G}(z)$ is AR, then polynomial matrix $U\boldsymbol{G}(z)\boldsymbol{W}(z)$ is also AR for any $N \times N$ nonsingular constant matrix U and any irreducible polynomial matrix $\boldsymbol{W}(z)$.

The AR precoders have been generalized in [23] to *polynomial ambiguity resistant* (PAR) precoders, for resisting not only constant matrix ambiguities, but also polynomial matrix ambiguities. The main advantage of using PAR precoders in the systems in Figs. 1 and 2 is that one can directly identify the input signals from the output signals by resolving the channel polynomial ambiguities without using any MIMO system identification algorithm. In the rest of this paper, however, for simplicity we focus on AR precoders, although an analogous approach applies to PAR precoders. Notice that the family in (2.2) and Lemma 1 is the only one known so far. It is, however, important to know more families of AR precoders and even more important to know criterions for justifying AR precoders and the optimal AR precoders. The following sections are devoted to the study of these problems.

III. More Properties and Families of AR Precoders

In this section, we want to present more properties and families of AR precoders. By Lemma 1, every known AR precoder $\boldsymbol{G}(z)$ can be used to generate a class of AR precoders by simply left and right multiplying it with nonsingular constant matrices and irreducible polynomial matrices, respectively. Clearly, the precoder in (2.2) generates a class of AR precoders. The question to ask here is whether there exist other AR precoder classes (or families) that are not generated by the one in (2.2). To study this question, we introduce *equivalent classes* for AR precoders.

Definition 1: Two $N \times K$ ambiguity resistant precoders $\boldsymbol{G}_1(z)$ and $\boldsymbol{G}_2(z)$ are in the same equivalent class if and only if there exist an $N \times N$ nonsingular constant matrix U and a $K \times K$ irreducible polynomial matrix $\boldsymbol{W}(z)$ such that

$$\boldsymbol{G}_2(z) = U\boldsymbol{G}_1(z)\boldsymbol{W}(z). \tag{3.1}$$

When precoders $\boldsymbol{G}_1(z)$ and $\boldsymbol{G}_2(z)$ are in the same equivalent class, they are called equivalent.

As a remark, the above equivalence is only for the AR property, i.e., if one is AR and then the other is also AR when they are equivalent, and does not mean the performance equivalence. Examples will be shown in Section V for the performance difference.

It is known that any polynomial matrix can be diagonalized by employing the following three elementary row and column operations:

1) interchange two rows (or columns);
2) multiply a row (or column) with a nonzero constant;
3) add a polynomial multiple of a row (or column) to another row (or column).

This diagonalization is called the Smith form decomposition (see for example [42]). Any of the above elementary operations can be realized by multiplying a certain matrix to the polynomial matrix at the left or the right side. In particular, the above 1) and 2) operations correspond to two nonsingular constant matrices and all operations 1)–3) correspond to irreducible polynomial matrices. This means that doing row operations 1) or 2), or any column operations of an AR precoder, does not change the AR property. Similar to the Smith form decomposition, the following result is not difficult to see.

Theorem 1: Any $N \times K$ ambiguity resistant polynomial matrix $\boldsymbol{G}(z)$ is equivalent to

$$\begin{bmatrix} g_{1,1}(z) & 0 & 0 & \cdots & 0 \\ g_{2,1}(z) & g_{2,2}(z) & 0 & \cdots & 0 \\ g_{3,1}(z) & g_{3,2}(z) & g_{3,3}(z) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{K,1}(z) & g_{K,2}(z) & g_{K,3}(z) & \cdots & g_{K,K}(z) \\ \boldsymbol{F}_1(z) & \boldsymbol{F}_2(z) & \boldsymbol{F}_3(z) & \cdots & \boldsymbol{F}_K(z) \end{bmatrix} \tag{3.2}$$

where $\boldsymbol{F}_k(z)$ are $(N-K) \times 1$ polynomial matrices

$$\deg(g_{1,1}(z)) \le \deg(g_{2,2}(z)) \le \cdots \le \deg(g_{K,K}(z))$$

and

$$\deg(g_{i,j}(z)) < \deg(g_{i,i}(z)) \text{ for any } j < i$$

and $\deg(f(z))$ denotes the degree of polynomial $f(z)$ of z^{-1}.

By doing row operations 1)–2) and column operations 1)–3), it is not hard to see that the known precoder in (2.2) is equivalent to

$$\boldsymbol{F}(z) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ z^{-(N-1)(r+1)} & z^{-(N-2)(r+1)} & \cdots & z^{-(r+1)} \end{bmatrix}_{N\times(N-1)} \tag{3.3}.$$

In this case, all diagonal elements $g_{k,k}(z)$ in (3.2) are 1.

Notice that the diagonal matrix in the Smith form decomposition of an $N \times K$ irreducible polynomial matrix is always

$$\begin{bmatrix} I_K \\ 0_{(N-K)\times K} \end{bmatrix} \tag{3.4}$$

by using the irreducibility, where the polynomial elementary operation 3) for rows is used, which may, however, change the AR property for an AR polynomial matrix. This is exactly the reason why the form (3.2) is already the simplest form for an $N \times K$ AR precoder. Unlike the diagonal matrix of an AR precoder in its Smith form decomposition, the diagonal elements $g_{k,k}(z)$ in its corresponding form (3.2) may not be nonzero constants, or equivalently $g_{k,k}(z)$ may not be all 1. In other words, the form in (3.3) may not be true in general. The following are two examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \\ z^{-3} & z^{-2}+1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ z^{-1} & z^{-2} \\ z^{-2} & 1+z^{-3} \end{bmatrix} \tag{3.5}$$

which are AR but not equivalent to the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ g_{3,1}(z) & g_{3,2}(z) \end{bmatrix}$$

for any two polynomials $g_{3,1}(z)$ and $g_{3,2}(z)$ of variable z^{-1}. This implies that other equivalent classes beyond the known one in (2.2) or (3.3) exist.

We next want to characterize a subclass of all AR precoders, which are equivalent to the forms in (3.2) with all diagonal elements 1, i.e.,

$$g_{1,1}(z) = g_{2,2}(z) = \ldots = g_{K,K}(z) = 1.$$

In this case, due to the property $\deg(g_{i,j}(z)) < \deg(g_{i,i}(z))$ for $j < i$ in Theorem 1, (3.2) becomes

$$\begin{bmatrix} I_K \\ \boldsymbol{F}(z) \end{bmatrix} \tag{3.6}$$

where $\boldsymbol{F}(z)$ is an $(N-K) \times K$ polynomial matrix. The precoder in (3.6) is called a *systematic precoder*, which is analogous to systematic convolutional codes. The following result characterizes all systematic AR precoders with form (3.6) when $K=N-1$, i.e., the smallest bandwidth expansion and the most interesting case.

Theorem 2: An $N\times(N-1)$ *systematic precoder*

$$\boldsymbol{G}(z)=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\\ G_1(z) & G_2(z) & \cdots & G_{N-1}(z)\end{bmatrix}_{N\times(N-1)} \tag{3.7}$$

is ambiguity resistant if, and only if, it is equivalent to

$$\boldsymbol{F}(z)=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\\ F_1(z) & F_2(z) & \cdots & F_{N-1}(z)\end{bmatrix}_{N\times(N-1)} \tag{3.8}$$

where $F_k(z)$ are polynomials of z^{-1} such that

$$\deg(F_1(z))>\deg(F_2(z))>\cdots>\deg(F_{N-1}(z))\geq 1. \tag{3.9}$$

Proof: By doing constant elementary operations 1)–3) and general polynomial column operations 1)–3), it is not hard to see that $\boldsymbol{G}(z)$ in (3.7) is equivalent to $\boldsymbol{F}(z)$ in (3.8), with $\deg(F_1(z))>\deg(F_2(z))>\cdots>\deg(F_{N-1}(z))\geq 0$. To prove the necessity, we only need to show $\deg(F_{N-1}(z))\neq 0$. Assume this is not true. Then $F_{N-1}(z)$ is a constant. This does not satisfy the necessary condition 1) in Lemma 1 for an AR precoder.

We next prove the sufficiency. Let $E=(e_{ij})_{N\times N}$ and $\boldsymbol{V}(z)=(v_{ij}(z))_{N\times N}$ such that $E\boldsymbol{G}(z)=\boldsymbol{G}(z)\boldsymbol{V}(z)$, i.e.,

$$\begin{bmatrix}e_{11} & \cdots & e_{1N}\\ \cdots & \cdots & \cdots\\ e_{N1} & \cdots & e_{NN}\end{bmatrix}\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\\ F_1(z) & F_2(z) & \cdots & F_{N-1}(z)\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\\ F_1(z) & F_2(z) & \cdots & F_{N-1}(z)\end{bmatrix}\cdot\begin{bmatrix}v_{11}(z) & \cdots & v_{1(N-1)}(z)\\ \cdots & \cdots & \cdots\\ v_{(N-1)1}(z) & \cdots & v_{(N-1)(N-1)}(z)\end{bmatrix}$$

we obtain

$$e_{ij}+e_{iN}F_j(z)=v_{ij}(z),\ \text{for } i,\,j=1,\,2,\,\ldots,\,N-1 \tag{3.10}$$

$$e_{Nj}+e_{NN}F_j(z)=\sum_{k=1}^{N-1}v_{kj}(z)F_k(z),\ \text{for } j=1,\,2,\,\ldots,\,N-1. \tag{3.11}$$

First, from (3.10) and (3.11), we obtain

$$e_{Nj}+e_{NN}F_j(z)=(e_{1j}+e_{1N}F_1(z))F_1(z)+\sum_{k=2}^{N-1}(e_{kj}+e_{kN}F_j(z))F_k(z).$$

Taking $j=1$, we have

$$e_{N1}+e_{NN}F_1(z)=(e_{11}+e_{1N}F_1(z))F_1(z)+\sum_{k=2}^{N-1}(e_{k1}+e_{kN}F_1(z))F_k(z).$$

By comparing the coefficients of the two polynomials, we have $e_{kN}=0$ for any $k=1,\ldots,N-1$. Hence, $v_{ij}(z)=e_{ij}$ is in fact a constant for any $i,\,j=1,\ldots,N-1$. Since $\deg(F_1(z))>\deg(F_2(z))>\cdots>\deg(F_{N-1}(z))\geq 1$, $F_1(z),\ldots,F_{N-1}(z)$ are linearly independent. By (3.11), again we have

$$e_{Nj}+e_{NN}F_j(z)=\sum_{k=1}^{N-1}v_{kj}F_k(z)\ \text{for } j=1,\,2,\,\ldots,\,N-1.$$

Hence $v_{ij}=e_{ij}=0$ except possibly $v_{jj}=e_{jj}=e_{NN}$ for $j=1,\,2,\,\ldots,\,N-1$. Since E is a nonzero matrix, $\boldsymbol{V}(z)=\alpha I_{N-1}$ for some nonzero constant α. This proves the sufficiency. □

After we characterize all AR precoders with their equivalent systematic forms in (3.7), we now want to specify all different equivalent classes of (3.8). By doing elementary operations, the AR precoders in (3.8) can be further simplified as follows.

Corollary 1: An $N\times(N-1)$ precoder $\boldsymbol{G}(z)$ in (3.7) is AR if and only if it is equivalent to

$$\boldsymbol{F}(z)=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\\ F_1(z) & F_2(z) & \cdots & F_{N-1}(z)\end{bmatrix}_{N\times(N-1)} \tag{3.12}$$

where

$$F_k(z)=\sum_{l=1}^{r_k}a_{kl}z^{-n_{kl}},\qquad k=1,\,2,\,\ldots,\,N-1 \tag{3.13}$$

where $r_k>0$ and $a_{kl}\neq 0$ when $r_k>1$

$$n_{k1}>n_{k2}>\cdots>n_{kr_k}\geq 0$$

and

$$n_{11}>n_{21}>\cdots>n_{(N-1)1}>0 \tag{3.14}$$

and the powers in $F_k(z)$ do not include any leading power $z^{-n_{p1}}$ in $F_p(z)$ for $p=k+1,\ldots,N-1$, i.e., the integer set $\{n_{kl}\colon l=2,\ldots,r_k\}$ does not intersect with the integer set $\{n_{l1}\colon l=k+1,\ldots,N-1\}$ for $k=1,\,2,\,\ldots,\,N-2$.

All AR precoders in (3.7) can be classified as follows.

Corollary 2: Two AR precoders in (3.7) are equivalent if and only if they can be equivalently reduced to the same form (3.12) in Corollary 1.

From Corollary 2, one can clearly see that there are many equivalent classes that are not the same as the one in (2.2) or (3.3).

IV. Optimal AR Precoders

In Section III, we have found several families of AR precoders. Although all AR precoders are good enough in theory to be used to cancel the ISI without additive noise, AR precoders

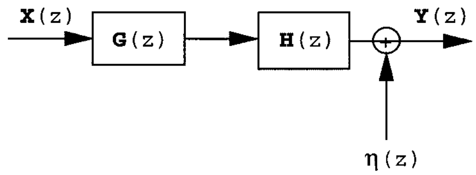

Fig. 3. Unified MIMO system.

may have performance difference when there is additive noise in the channel. Then the question becomes which AR precoder is "better," where "better" means better symbol error rate performance at the receiver after equalization. In this section, we study a criterion for AR precoders and also optimal AR precoders by introducing the distance concept for a precoder.

A. Distance and Criterion for AR Precoders

To study the above question, let us briefly recall the conventional error-control coding theory, see for example [43]. In error-control coding, inputs, code coefficients and outputs are all in a finite field, such as zero and one, and the coding arithmetic is the finite field arithmetic. Therefore, the "distance" concept is clear, for example the Hamming distance between two finite sequences of 0s and 1s. Moreover, the minimum distance between all coded sequences can be calculated from the code itself. The minimum distance controls the performance of the error rate at the receiver for decoded sequences, when only additive random noise occurs in the channel. Code A is better than Code B if the minimum distance of Code A is larger than the minimum distance of Code B. The *key* for this criterion and the above concepts to hold is that all inputs, code coefficients and outputs are in the same *finite* field and there is only additive random noise in the channel. This does not hold in the precoding studied in Sections II and III, where at first inputs, precoder coefficients and outputs are all in the complex-valued field and then the channel has ISI besides additive random noise. Although this is the case, the "distance" of the ISI channel output values also controls the performance in resisting additive channel random noise. To the first issue, the conventional Hamming distance does not apply, and the Euclidean distance for the output signal values after precoding needs to be used. Since it is hard to deal with the minimum Euclidean distance concept in the complex-valued field, the Euclidean distance here is in the mean sense when the input signal is modeled as a complex-valued random process. To the second issue, we need to investigate how the Euclidean distance of the output values of a precoder affects the Euclidean distance of the output values of the ISI channel, which determines the performance of the precoder in resisting additive random noise.

To study these issues, let us go back to the systems with ISI in Figs. 1 and 2. By blocking the ISI channels from serial to parallel, the systems in Figs. 1–2 can be unified into the MIMO system shown in Fig. 3, where $\boldsymbol{X}(z)$ is the $K \times 1$ polynomial matrix of the z-transform of the input vectors, $\boldsymbol{G}(z)$ is the $N \times K$ AR precoder, $\boldsymbol{H}(z)$ is the $M \times N$ polynomial matrix of the ISI channel, $\eta(z)$ is the $M \times 1$ polynomial matrix of the z-transform of the additive white noise vectors, and $\boldsymbol{Y}(z)$ is the $M \times 1$ polynomial matrix of the z-transform of the channel output vectors. Let

$$\boldsymbol{G}(z) = \sum_{n=0}^{Q_G} G(n) z^{-n}, \quad \boldsymbol{H}(z) = \sum_{n=0}^{Q_H} H(n) z^{-n}$$
$$\boldsymbol{X}(z) = \sum_n X(n) z^{-n}, \quad \boldsymbol{Y}(z) = \& \sum_n Y(n) z^{-n}.$$

Let

$$\boldsymbol{V}(z) \triangleq \boldsymbol{G}(z)\boldsymbol{X}(z) = \sum_n V(n) z^{-n}$$

the z-transform of the precoder output vector sequence, and

$$\boldsymbol{U}(z) \triangleq \boldsymbol{H}(z)\boldsymbol{V}(z) = \sum_n U(n) z^{-n}$$

the z-transform of the ISI channel output vector sequence. Notice that all $X(n)$, $Y(n)$, $V(n)$, $V(n)$, $\eta(n)$ are constant column vectors, while $G(n)$, $H(n)$ are constant matrices. To study the mean distance for the output values in $U(n)$, let us use matrix representations for linear transformations. By concatenating all vectors $X(n)$ together, all vectors $V(n)$ together, all vectors $U(n)$ together, all vectors $\eta(n)$ together, and all vectors $Y(n)$ together, we obtain bigger block vectors $\mathcal{X} = (x(n))$, $\mathcal{V} = (v(n))$, $\mathcal{U} = (u(n))$, $\eta = (\eta(n))$, and $\mathcal{Y} = (y(n))$, respectively. Let $\mathcal{G}$ and $\mathcal{H}$ denote the generalized Sylvester matrices, respectively

$$\mathcal{G} = \begin{bmatrix} G(Q_G) & \cdots & G(0) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & G(Q_G) & \cdots & G(0) \end{bmatrix}$$
$$\mathcal{H} = \begin{bmatrix} H(Q_H) & \cdots & H(0) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & H(Q_H) & \cdots & H(0) \end{bmatrix}. \quad (4.1)$$

Then

$$\mathcal{V} = \mathcal{G}\mathcal{X}, \quad \mathcal{U} = \mathcal{H}\mathcal{V}, \quad \mathcal{Y} = \mathcal{U} + \eta. \quad (4.2)$$

In what follows, for convenience we assume the input signal $x(n)$ is an i.i.d. random process with mean zero and variance σ_x^2. Thus, random processes $v(n)$ and $u(n)$ have zero mean. We also assume all coefficients in the ISI channel $\boldsymbol{H}(z)$ are i.i.d. with mean zero and variance σ_H^2 and are independent of $x(n)$. Notice that this assumption is only used to simplify the following analysis and does not apply to the single receiver system in Fig. 1, where the corresponding channel matrix $\boldsymbol{H}(z)$ has the pseudo-circulant structure [42].

The mean distances between all values of $u(n)$ and all values of $v(n)$ are

$$d_v \triangleq \left(E\left(\sum_{m,n} |v(m) - v(n)|^2 \right) \right)^{1/2}$$
$$d_u \triangleq \left(E\left(\sum_{m,n} |u(m) - u(n)|^2 \right) \right)^{1/2} \quad (4.3)$$

respectively, where E means the expectation. By the assumptions on the coefficients of $\boldsymbol{H}(z)$, it is not hard to see the following relationship between the mean distance d_u of the ISI channel output values $u(n)$ and the mean distance d_v of the precoder output values (or the ISI channel input values) $v(n)$:

$$d_u = \sigma_H d_v. \tag{4.4}$$

This can be stated in the following lemma.

Lemma 3: The performance of a precoder in resisting additive channel white noise is proportional to the mean distance of the precoder output values.

The above result solves the second issue that arose in the beginning of this section, and we need only to study the mean distance d_v of all the precoder output values for the performance of resisting additive channel random errors. Based on the above analysis, we have the following definition for optimal AR precoders.

Defintion 2: An $N \times K$ ambiguity resistant precoder $\boldsymbol{G}(z)$ is called optimal *if the mean distance d_v of all the precoder output values is the maximal among all $N \times K$ ambiguity resistant precoders, when the total energy is fixed.*

The squared mean distance d_v can be calculated as

$$\begin{aligned} d_v^2 &= \sum_{m,\,n} E|v(m) - v(n)|^2 \\ &= 2(LN-1)\sum_n E(|v(n)|^2) - 2\sum_{m\neq n} E(v(m)v^*(n)) \end{aligned} \tag{4.5}$$

where L is the length of the precoder output vector sequence $V(n)$, and N is the precoder size. Let $R(m, n)$ be the correlation function of the random process $v(n)$, i.e.,

$$R(m, n) = E(v(m)v^*(n)).$$

Let $\mathcal{R}$ be the correlation matrix of $v(n)$, i.e.,

$$\mathcal{R} = (R(m, n)) = E(\mathcal{G}\mathcal{X}(\mathcal{G}\mathcal{X})^\dagger) = \mathcal{G}E(\mathcal{X}\mathcal{X})^\dagger\mathcal{G}^\dagger = \sigma_x^2\mathcal{G}\mathcal{G}^\dagger \tag{4.6}$$

where † means the conjugate transpose. One can see that the first and second term on the right-hand side of (4.5) for the distance d_v are the sum of all the diagonal elements, i.e., the trace of the matrix $\mathcal{G}\mathcal{G}^\dagger$ multiplied by $2\sigma_x^2$, and the sum of all the off-diagonal elements of the matrix $\mathcal{G}\mathcal{G}^\dagger$ multiplied by $2\sigma_x^2$, respectively. In formula, the squared mean distance d_v can be calculated as

$$\begin{aligned} d_v^2 &= 2\sigma_x^2\left((LN-1)\text{trace}(\mathcal{G}\mathcal{G}^\dagger) - \sum_{m\neq n}(\mathcal{G}\mathcal{G}^\dagger)_{mn}\right) \\ &= 2\sigma_x^2\left(LN\text{trace}(\mathcal{G}\mathcal{G}^\dagger) - \sum_{m,\,n}(\mathcal{G}\mathcal{G}^\dagger)_{mn}\right) \end{aligned} \tag{4.7}$$

where $(\mathcal{G}\mathcal{G}^\dagger)_{mn}$ denotes the element at the mth row and the nth column of $\mathcal{G}\mathcal{G}^\dagger$.

We next want to simplify d_v in (4.7) by using all the coefficients in the precoder $\boldsymbol{G}(z)$. For a precoder $\boldsymbol{G}(z)$, define

$$D_G \triangleq \text{sum of all coefficients of all coefficient matrices of } \boldsymbol{G}(z)\boldsymbol{G}^\dagger(1/z) \tag{4.8}$$

$$E_G \triangleq \text{sum of all magnitude squared coefficients of all coefficient matrices of } \boldsymbol{G}(z) \tag{4.9}$$

where $\boldsymbol{G}^\dagger$ means the conjugate transpose of all coefficient matrices of $\boldsymbol{G}(z)$. Let L be the length of the precoder output vector sequence $V(n)$. Then, by (4.1), it is not hard to see that

$$\text{trace}(\mathcal{G}\mathcal{G}^\dagger) = LE_G, \quad \sum_{m,\,n}(\mathcal{G}\mathcal{G}^\dagger)_{mn} = LD_G. \tag{4.10}$$

Therefore

$$d_v^2 = 2\sigma_x^2 L(LNE_G - D_G). \tag{4.11}$$

Since E_G is fixed as the total energy of all the coefficients of the coefficient matrices in $\boldsymbol{G}(z)$, and σ_x^2, L, and N are also fixed, based on formula (4.11) for the mean distance d_v, we have the following criterion for judging the performance of an AR precoder.

Defintion 3: $N \times K$ ambiguity resistant precoder $\boldsymbol{G}(z)$ is said better *than $N \times K$ ambiguity resistant precoder $\boldsymbol{F}(z)$ if $D_G < D_F$ when $E_G = E_F$, where D_G, D_F, E_G, and E_F are defined by (4.8) and (4.9) for precoders $\boldsymbol{G}(z)$ and $\boldsymbol{F}(z)$, respectively*.

Based on formula (4.11) on the mean distance d_v of the precoder output values, we define the distance for a precoder as follows.

Defintion 4: For an $N \times K$ precoder $\boldsymbol{G}(z)$, its distance *is defined by*

$$d(G) \triangleq N - \frac{D_G}{E_G}$$

where D_G and E_G are defined in (4.8) and (4.9).

With the above two definitions, the following corollary is straightforward.

Corollary 3: AR precoder $\boldsymbol{G}(z)$ is better than AR precoder $\boldsymbol{F}(z)$ if, and only if, the distance of $\boldsymbol{G}(z)$ is greater than the distance of $\boldsymbol{F}(z)$, i.e., $d(G) > d(F)$.

Let us see two examples. Consider the two AR precoders $\boldsymbol{G}(z)$ in (2.2) and $\boldsymbol{F}(z)$ in (3.3). It is not hard to see that $E_G = E_F = 2(N-1)$, and when $N > 2$, we have

$$\boldsymbol{G}(z)\boldsymbol{G}^\dagger(1/z) = \begin{bmatrix} 1 & z^{r+1} & 0 & \cdots & 0 & 0 & 0 \\ z^{-r-1} & 2 & z^{r+1} & \cdots & 0 & 0 & 0 \\ 0 & z^{-r-1} & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-r-1} & 2 & z^{r+1} \\ 0 & 0 & 0 & \cdots & 0 & z^{-r-1} & 1 \end{bmatrix}_{N\times N}$$

and the equation at the bottom of the page. Thus, when $N > 2$, $D_G = D_F = 4(N-1)$. Therefore, when $N > 2$

$$d(G) = d(F) = N - \frac{4(N-1)}{2(N-1)} = N - 2.$$

This proves the following corollary.

Corollary 4: The AR precoders $\boldsymbol{F}(z)$ in (3.3) and $\boldsymbol{G}(z)$ in (2.2) have distances $d(G) = d(F) = N - 2$. In other words, these two precoders are equivalent not only with respect to the AR property, but also with respect to the distance property.

This corollary will be seen by the numerical simulations in Section V, where QPSK is used for the input signals and there is only a small performance difference between these two precoders. This small performance difference is caused from the specific QPSK modulation scheme, as we will see in Section V, and the above result holds in general.

Since the precoder output vector length L, the precoder size N, and the input signal variance σ_x^2 are fixed, the following theorem is straightforward from (4.11).

Theroem 3: An $N \times K$ ambiguity resistant precoder $\boldsymbol{G}(z)$ is optimal in all $N \times K$ ambiguity resistant precoders if and only if the total sum D_G of all the coefficients of all the coefficient matrices of the product matrix $\boldsymbol{G}(z)\boldsymbol{G}^\dagger(1/z)$ is minimal among all possible $N \times K$ ambiguity resistant precoders $\boldsymbol{F}(z)$ when the total sum E_G of all the magnitude squared coefficients of all coefficient matrices of $\boldsymbol{F}(z)$ is fixed.

We now want to find a family of column operations of a precoder so that they do not change the distance property. A $K \times K$ polynomial matrix $\boldsymbol{U}(z)$ is called *paraunitary* if, and only if

$$\boldsymbol{U}(z)\boldsymbol{U}^\dagger(1/z) = I_K.$$

For more about paraunitary matrices, see [42], where all paraunitary matrices are characterized by using the lattice factorization. In [42], instead of using $\boldsymbol{U}^\dagger(1/z)$, the tilde operation notation $\tilde{\boldsymbol{U}}(z)$ is used. With paraunitary polynomial matrices, we have the following result.

Corollary 5: Let $\boldsymbol{U}(z)$ be a $K \times K$ paraunitary matrix. If $\boldsymbol{G}(z)$ is an $N \times K$ AR precoder with distance $d(G)$, then $\boldsymbol{G}(z)\boldsymbol{U}(z)$ is also an AR precoder with distance $d(G)$, i.e., $d(GU) = d(G)$. If $\boldsymbol{G}(z)$ is an optimal $N \times K$ AR precoder, then so is $\boldsymbol{G}(z)\boldsymbol{U}(z)$.

Proof: From (4.8), clearly $D_G = D_{GU}$. Since the sum of all magnitude squared coefficients of all coefficient matrices of $\boldsymbol{G}(z)$ is equal to the sum of all diagonal elements of the coefficient matrix of the constant term z^{-0} in the matrix $\boldsymbol{G}(z)\boldsymbol{G}^\dagger(1/z)$ and $\boldsymbol{G}(z)\boldsymbol{U}(z)\boldsymbol{U}^\dagger(1/z)\boldsymbol{G}^\dagger(1/z) = \boldsymbol{G}(z)\boldsymbol{G}^\dagger(1/z)$, we have $E_G = E_{GU}$. Thus, by Definition 4, we have $d(G) = d(GU)$. □

Notice that

$$\begin{aligned} \sigma_x^2 L D_G &= \sigma_x^2 \sum_{m,n} (\mathcal{G}\mathcal{G}^\dagger)_{mn} \\ &= \sum_{m,n} E(v(m)v^*(n)) \\ &= E\left|\sum_n v(n)\right|^2 \geq 0. \end{aligned} \tag{4.12}$$

Using (4.11), the following upper bound for the mean distance d_v is proven.

Theorem 4: The mean distance d_v of the precoder output values for an $N \times K$ precoder $\boldsymbol{G}(z)$ is upper bounded by

$$d_v \leq \sigma_x L\sqrt{2N}\sqrt{E_G} \tag{4.13}$$

where σ_x^2 is the input signal variance, L is the length of the precoder output vector sequence, and E_G is defined by (4.9), i.e., the total energy of all coefficients in $\boldsymbol{G}(z)$. The upper bound for the distance of an $N \times K$ precoder $\boldsymbol{G}(z)$ is $d(G) \leq N$.

Now the question is: can the above upper bound be reached? Clearly the precoders that reach the upper bound in (4.13) are optimal. In the next subsection, we shall answer this question positively. Notice that, when there is no precoding, i.e., $\boldsymbol{G}(z) = I_K$, we have

$$E\left|\sum_n v(n)\right|^2 = \sum_n E|v(n)|^2 = \sum_n E|x(n)|^2 = LK\sigma_x^2 > 0.$$

The mean distance of the precoder output values is

$$d_v = \left(\sum_{m,n} E|v(m) - v(n)|^2\right)^{1/2} = \sigma_x\sqrt{2(LK-1)LK}$$

and the precoder distance is $d(G) = K - 1 = N - 1$ in this case.

B. Optimal Systematic AR Precoders

In this subsection, we determine all optimal systematic AR precoders by using the criterion proposed in Section IV-A. We have the following result.

Theroem 5: An $N \times (N-1)$ systematic ambiguity resistant precoder $\boldsymbol{F}(z)$ in (3.8) with

$$F_k(z) = \sum_{l=0}^{n_k} a_{kl} z^{-l}, \quad a_{kn_k} \neq 0, \qquad 1 \leq k \leq N-1$$
$$n_1 > n_2 > \cdots > n_{N-1} \geq 1 \tag{4.14}$$

is optimal if, and only if

$$\sum_{l=0}^{n_k} a_{kl} = -1, \text{ for } k = 1, 2, \ldots, N-1. \tag{4.15}$$

Moreover, for the above optimal precoder, the mean distance d_v of the precoder output values and the precoder distance $d(F)$ are

$$d_v = \sigma_x L\sqrt{2N}\sqrt{E_F}, \quad d(F) = N \tag{4.16}$$

$$\boldsymbol{F}(z)\boldsymbol{F}^\dagger(1/z) = \begin{bmatrix} 1 & 0 & \cdots & 0 & z^{(N-1)(r+1)} \\ 0 & 1 & \cdots & 0 & z^{(N-2)(r+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & z^{r+1} \\ z^{-(N-1)(r+1)} & z^{-(N-2)(r+1)} & \cdots & z^{-(r+1)} & N-1 \end{bmatrix}_{N\times N}$$

where σ_x^2 is the variance of the input signal, L is the length of the precoder output vector sequence, and

$$E_F = N - 1 + \sum_{k=1}^{N-1} \sum_{l=0}^{n_k} |a_{kl}|^2. \tag{4.17}$$

Proof: By (4.9), E_F is clearly the total sum of all the coefficients in all coefficient matrices of the precoder $\boldsymbol{F}(z)$. To calculate D_F in (4.8) for $\boldsymbol{F}(z)$, the product matrix $\boldsymbol{F}(z)\boldsymbol{F}^\dagger(1/z)$ is

$$\boldsymbol{F}(z)\boldsymbol{F}^\dagger(1/z) = \begin{bmatrix} 1 & 0 & \cdots & 0 & F_1^*(1/z) \\ 0 & 1 & \cdots & 0 & F_2^*(1/z) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & F_{N-1}^*(1/z) \\ F_1(z) & F_2(z) & \cdots & F_{N-1}(z) & F_0(z) \end{bmatrix}_{N\times N}$$

where

$$F_k^*(1/z) = \sum_{l=0}^{N-1} a_{kl}^* z^l$$

and

$$\begin{aligned} F_0(z) &= \sum_{k=1}^{N-1} F_k(z) F_k^*(1/z) \\ &= \sum_{k=1}^{N-1} \sum_{l_1=0}^{n_k} \sum_{l_2=0}^{n_k} a_{kl_1} a_{kl_2}^* z^{-(l_1-l_2)}. \end{aligned}$$

Thus, it is not hard to see that

$$\begin{aligned} D_F &= N - 1 + \sum_{k=1}^{N-1} \sum_{l=0}^{n_k} (a_{kl} + a_{kl}^*) + \sum_{k=1}^{N-1} \sum_{l_1=0}^{n_k} \sum_{l_2=0}^{n_k} a_{kl_1} a_{kl_2}^* \\ &= \sum_{k=1}^{N-1} \left| \sum_{l=0}^{n_k} a_{kl} + 1 \right|^2. \end{aligned}$$

Therefore, the minimum of D_F over all $\boldsymbol{F}(z)$ in (4.14) is reached if, and only if, $D_F = 0$. In other words, D_F is minimal if, and only if, (4.15) holds, where E_F in (4.17) is fixed.

When $D_F = 0$, i.e., the precoder $\boldsymbol{F}(z)$ is optimal, the optimal mean distance formula (4.16) for the precoder $\boldsymbol{F}(z)$ follows from (4.11). □

This theorem also implies that there exist AR precoders that reach the upper bound (4.13), i.e., $D_G = 0$. By (4.12), the following corollary is straightforward.

Corollary 6: The following statements are equivalent.

1) *An $N \times K$ AR precoder $\boldsymbol{G}(z)$ is optimal.*
2) *$D_G = 0$, i.e., the total sum of all coefficients of all coefficient matrices of $\boldsymbol{G}(z)\boldsymbol{G}^\dagger(1/z)$ is zero.*
3) *The distance of the precoder $\boldsymbol{G}(z)$ is $d(G) = N$.*

By the above results and Corollary 4, the precoders $\boldsymbol{G}(z)$ in (2.2) and $\boldsymbol{F}(z)$ in (3.3) are not optimal. The precoder in [20] and [21]

$$\begin{bmatrix} I_K \\ 0_{(N-K)\times K} \end{bmatrix}$$

is not optimal either, which is not AR and its distance is zero.

Given size N, the simplest optimal $N \times (N-1)$ systematic AR precoders are

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ -z^{-n_1} & -z^{-n_2} & \cdots & -z^{-n_{N-1}} \end{bmatrix}_{N\times(N-1)} \tag{4.18}$$

where $n_1 > n_2 > \cdots > n_{N-1} \geq 1$.

V. Numerical Examples

In this section, we want to present some examples to illustrate the theory obtained in the previous sections. Since all numerical simulations in this section are only used to prove the concepts in resisting additive channel random errors, some simplifications are made. These simplifications include that an MIMO system identification algorithm has been implemented, i.e., there is only a nonsingular constant matrix ambiguity in the ISI channel.

We consider the undersampled communication system in Fig. 2 with five antennas, and downsampling by factor 4. After a MIMO system identification algorithm is implemented, the ISI channel matrix becomes a 4×4 nonsingular constant matrix. Thus, we simply assume the ISI channel matrix as a 4×4 nonsingular constant matrix, and then a white noise $\eta(n)$ is added to the ISI channel output, as shown in Fig. 4(a). Notice that the 4×4 ISI channel constant matrix corresponds four antenna array receivers, where each channel has four-tap ISI by using the interpretation of the combination of the polyphase components [42], as shown in Fig. 4(b).

We now consider the following five 4×3 AR precoders:

$$\boldsymbol{G}_1(z) = \begin{bmatrix} 1 & 0 & 0 \\ z^{-1} & 1 & 0 \\ 0 & z^{-1} & 1 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$\boldsymbol{G}_2(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^{-3} & z^{-2} & z^{-1} \end{bmatrix}$$

$$\boldsymbol{G}_3(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}}(z^{-3}+z^{-2}) & \frac{1}{\sqrt{2}}(z^{-2}+z^{-1}) & \frac{1}{\sqrt{2}}(z^{-1}+1) \end{bmatrix}$$

$$\boldsymbol{G}_4(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -z^{-3} & -z^{-2} & -z^{-1} \end{bmatrix}$$

$$\boldsymbol{G}_5(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ az^{-3} + bz^{-2} & cz^{-2} + dz^{-1} & -z^{-1} \end{bmatrix}$$

where

$$a = -\frac{\sqrt{3}+3}{4}, \quad b = \frac{\sqrt{3}-1}{4}$$
$$c = -\frac{\sqrt{3}+1}{4}, \quad d = \frac{\sqrt{3}-3}{4}.$$

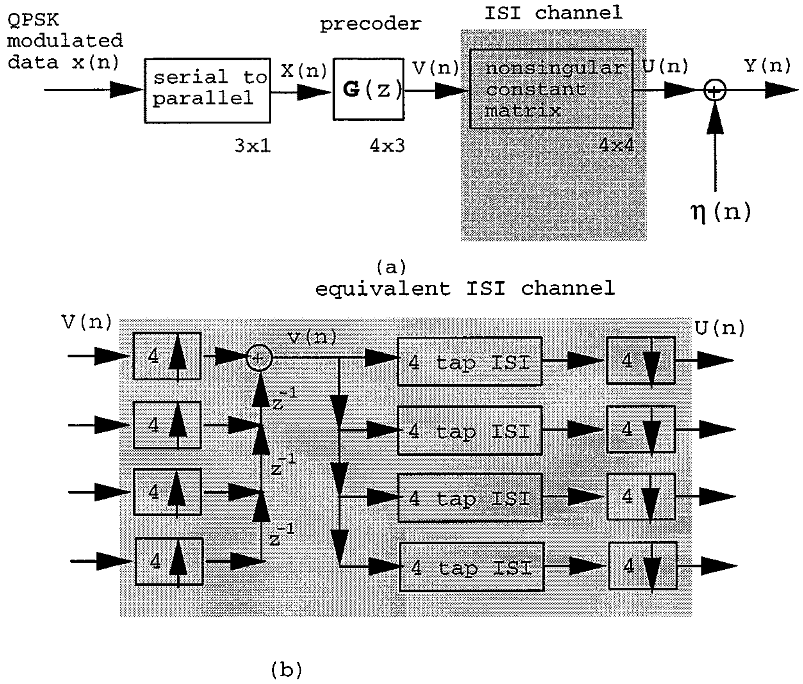

Fig. 4. Simplified undersampled antenna receiver system.

By Corollary 2, they are all in the same equivalent class. By Theorem 5, the precoders $\boldsymbol{G}_4(z)$ and $\boldsymbol{G}_5(z)$ are optimal. All $E_{G_i} = 6$ for $i = 1, 2, \ldots, 5$ for all these precoders. Their distances are

$$d(G_1) = d(G_2) = 4 - 2 = 2 \text{ by Corollary 4}$$
$$d(G_4) = d(G_5) = 4 \text{ by Theorem 5}$$

and

$$d(G_3) = 4 - \frac{3(\sqrt{2}+1)^2}{6} = \frac{5 - 2\sqrt{2}}{2} \approx 1.0858.$$

QPSK modulation is used for the input signal of the precoder. The *linear* closed-form equalization algorithm developed in [22]–[23] is used for the decoding. For more about closed-form blind equalization, see for example [28]–[30]. Three-hundred Monte Carlo iterations are used. Fig. 5 shows the QPSK symbol error rate comparison of these five precoders via the SNR for the additive channel white noise. Clearly, the two optimal precoders $\boldsymbol{G}_4(z)$ and $\boldsymbol{G}_5(z)$ outperform the other nonoptimal precoders $\boldsymbol{G}_i(z)$ for $i = 1, 2, 3$. Since $d(G_1) = d(G_2)$, theoretically these two precoders should have the same symbol error rate performance. From Fig. 5, one can see that the performance difference between these two precoders is small, where the small difference is purely due to the specific QPSK modulation. The theoretical result obtained in Section IV holds for general modulation schemes as mentioned earlier.

VI. Conclusion

In this paper, we presented more families and properties of AR precoders for ISI/multipath cancellation. We introduced the concept of *equivalent classes* for AR precoders and characterized all $N \times (N-1)$ systematic AR precoders. Many equivalent classes of AR precoders beyond the known one presented in [22] and [23] were found.

More importantly, we introduced the concepts of precoder *distance* and *optimal* AR precoders in justifying an AR precoder. Given an $N \times K$ precoder $\boldsymbol{G}(z)$, its distance is defined by $d(G) = N - D_G/E_G$, where D_G is the total sum of all coefficients of all coefficient matrices of the matrix $\boldsymbol{G}(z)\boldsymbol{G}^{\dagger}(1/z)$ and E_G is the total sum of all magnitude squared coefficients of all coefficient matrices of the matrix $\boldsymbol{G}(z)$. With this distance definition, an $N \times K$ AR precoder is optimal if, and only if, its distance is N. Furthermore, we characterized all $N \times (N-1)$ optimal systematic AR precoders. With this characterization, one is able to construct all possible optimal $N \times (N-1)$ systematic AR precoders. Finally, numerical simulations were presented to illustrate the theory and the concepts. Our numerical examples showed that an optimal AR precoder has good performance in resisting both of the channel ISI and additive random noise. The theory developed in this paper applies to not only single transmitting antenna systems, but also multiple transmitting antenna systems, such as space–time precoding.

Notice that the precoding studied in this paper and also in [20]–[23] are after the symbol modulation is done, i.e., the input signal of the precoding is complex-valued. This presents a combination of the modulation (first) and the coding (second), while the traditional trellis coded modulation (TCM) is the combination of the coding (first) and the modulation (second). Since the coding (in finite field) takes place before the modulation (in the complex valued field), it is not easy to incorporate with the consideration of the channel ISI (in the complex valued field) cancellation. The study in this paper might suggest a new approach called modulated precoding (MPC) to resist both the ISI and the additive random noise. Notice that both MPC and TCM do not need to know the ISI channel characteristics, if there are no additional ISI cancellation methods are used. This brings up an

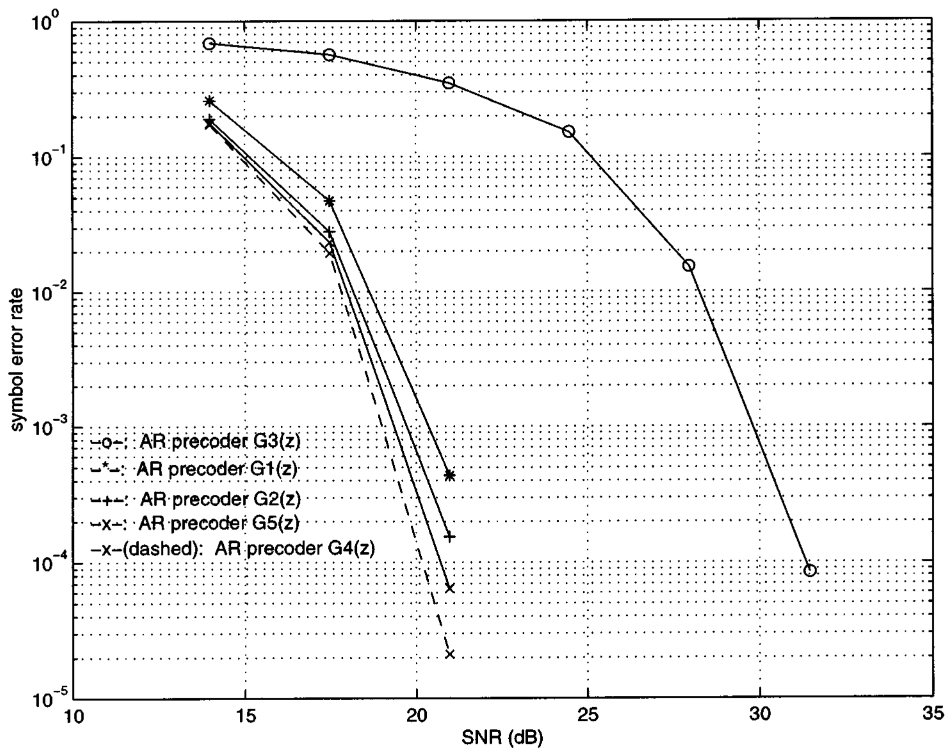

Fig. 5. Symbol error rate comparison. Solid line with $*$: $\boldsymbol{G}_1(z)$. Solid line with $+$: $\boldsymbol{G}_2(z)$. Solid line with O: $\boldsymbol{G}_3(z)$. Dashed line with $\times$: $\boldsymbol{G}_4(z)$. Solid line with $\times$: $\boldsymbol{G}_5(z)$.

interesting question: is MPC better than TCM? Two clear advantages of the MPC studied in this paper over TCM are: 1) the ISI cancellation capability because the precoding is designed for cancelling the ISI and 2) possible closed-form linear decoding algorithms, i.e., fast algorithms, because we are dealing with the same complex-valued field for all input and output signals and arithmetic and not a combination of a finite field and the complex valued field like in TCM. Further studies between the performance difference of these two different approaches, i.e., TCM and MPC, is under our current investigations. Notice that this paper has not taken a specific modulation scheme into an account, but provides the feasibility. The theory developed in this paper holds for a general modulation scheme. To optimally combine the modulation and the precoding in the MPC similar to TCM is another research topic in our investigations. More comprehensive studies in this direction can be found in [44].

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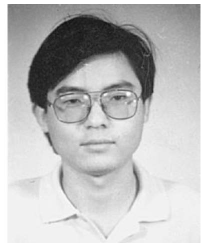

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