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LYM inequalities for t-antichains*

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Abstract Some LYM-type inequalities are derived for a class of special antichains called t-antichains, which has applications in unidirectional error detection codes.

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Let P be a partially ordered set with relation "$\leqslant$". An antichain is a subset F of P whose elements are totally unrelated; that is, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in F then $\boldsymbol{x} \not\leqslant \boldsymbol{y}$ and $\boldsymbol{y} \not\leqslant \boldsymbol{x}$. A chain opposed to an antichain is a subset E of P whose elements are totally related; that is, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in E, then either $\boldsymbol{x} \leqslant \boldsymbol{y}$ or $\boldsymbol{y} \leqslant \boldsymbol{x}$.

Example 1. Let $P = P_n = \{0, 1\}^n$; that is, each element $\boldsymbol{x}$ in P can be expressed as $\boldsymbol{x} = x_1 x_2 \cdots x_n$ with $x_i = 0$ or 1 for $i = 1, 2, \cdots, n$. The partial order "$\leqslant$" is defined as

$$\boldsymbol{x} \leqslant \boldsymbol{y} \text{ if and only if } \forall 1 \leqslant i \leqslant n, \ [x_i = 1 \Rightarrow y_i = 1], \ \forall \boldsymbol{x}, \ \boldsymbol{y} \in P_n, \tag{1}$$

where $\boldsymbol{x} = x_1 x_2 \cdots x_n$ and $\boldsymbol{y} = y_1 y_2 \cdots y_n$. If (1) is satisfied, we also say that $\boldsymbol{x}$ is covered by $\boldsymbol{y}$ or $\boldsymbol{y}$ covers $\boldsymbol{x}$. For example, $1100 \leqslant 1101$ with $n = 4$.

In what follows, $P = P_n = \{0, 1\}^n$, elements in P_n are called n-vectors denoted by letters, such as $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}$, $\boldsymbol{y}$ with components a_i, b_i, x_i, y_i, respectively. To prevent confusion in understanding, throughout this paper, for an n-vector $\boldsymbol{a} \in \{0, 1\}^n$, $|\boldsymbol{a}|$ denotes its weight; that is, the number of 1's in $\boldsymbol{a}$, and for a set $F \subset \{0, 1\}^n$, $|F|$ denotes its cardinality.

A maximal chain in P_n is a sequence of n-vectors: $\boldsymbol{0} = \boldsymbol{a}_0 \leqslant \boldsymbol{a}_1 \leqslant \boldsymbol{a}_2 \leqslant \cdots \leqslant \boldsymbol{a}_n$, where the weight $|\boldsymbol{a}_i|$ of $\boldsymbol{a}_i$ satisfies $|\boldsymbol{a}_i| = i$ for each i with $0 \leqslant i \leqslant n$. Then there are exactly $n!$ maximal chains in P_n, and exactly $k!(n-k)!$ maximal chains passing a given n-vector $\boldsymbol{a}$ of weight k. If F is an antichain in P_n, then each maximal chain contains at most one member of F. Therefore,

$$\sum_{\boldsymbol{a} \in F} |\boldsymbol{a}|! \, (n - |\boldsymbol{a}|)! \leqslant n! \, . \tag{2}$$

This gives us the following well-known LYM inequality[1-8] for antichain F:

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$$\sum_{\boldsymbol{a}\in F}\frac{1}{\binom{n}{|\boldsymbol{a}|}}\leqslant 1. \tag{3}$$

If we denote by f_k the number of n-vectors in F of weight k, then (3) becomes

$$\sum_{k=0}^{n}\frac{f_k}{\binom{n}{k}}\leqslant 1. \tag{4}$$

The LYM inequality was sharpened by Ahlswede and Zhang (see ref. [5]) to the Ahlswede-Zhang identity as follows. For any subset F of P_n, the A-Z identity is

$$\sum_{\boldsymbol{a}\in P_n}\frac{W_F(\boldsymbol{a})}{|\boldsymbol{a}|\binom{n}{|\boldsymbol{a}|}}\equiv 1, \tag{5}$$

where $W_F(\boldsymbol{a})=|\wedge_{\boldsymbol{b}:\boldsymbol{b}\leqslant\boldsymbol{a}\text{ and }\boldsymbol{b}\in F}\boldsymbol{b}|$.

In the definition of $W_F(\boldsymbol{a})$, $\boldsymbol{b}_1\wedge\boldsymbol{b}_2\triangleq\boldsymbol{c}$ is defined by $c_i=\min\{b_{1,i}, b_{2,i}\}$ where $\boldsymbol{c}=c_1c_2\cdots c_n$ and $\boldsymbol{b}_j=b_{j,1}b_{j,2}\cdots b_{j,n}$ for $j=1, 2$. For example, $11101\wedge 10110=10100$.

After we have the above well-known results, let us define what t-antichains are. To do so, we first have some notations.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two n-vectors, then $N(\boldsymbol{a}, \boldsymbol{b})$ is defined by

$$N(\boldsymbol{a}, \boldsymbol{b})\triangleq|\{1\leqslant i\leqslant n: a_i=1\wedge b_i=0\}|. \tag{6}$$

As an example, if $\boldsymbol{a}=10010$ and $\boldsymbol{b}=01011$, then $N(\boldsymbol{a}, \boldsymbol{b})=1$ and $N(\boldsymbol{b}, \boldsymbol{a})=2$. Obviously, the Hamming distance $d_H(\boldsymbol{a}, \boldsymbol{b})=N(\boldsymbol{a}, \boldsymbol{b})+N(\boldsymbol{b}, \boldsymbol{a})$. We now state the definition of t-antichains.

Definition 1. A subset $C\subset P_n$ is called a t-antichain if

$$\forall\boldsymbol{a}, \boldsymbol{b}\in C, \boldsymbol{a}\neq\boldsymbol{b}, [N(\boldsymbol{a}, \boldsymbol{b})\geqslant t+1]. \tag{7}$$

Example 2. $C=\{110000, 001100, 000011\}$ is a 1-antichain in P_6.

Obviously, a 0-antichain and an antichain are equivalent. Moreover, if C is a non-trivial t-antichain, that is, there are at least two n-vectors in C, then

$$\forall\boldsymbol{c}\in C, |\boldsymbol{c}|\geqslant t+1, n-|\boldsymbol{c}|\geqslant t+1, \text{ and } n\geqslant 2t+2. \tag{8}$$

Since a t-antichain C with $t>0$ is a special antichain, the question is whether the LYM inequality (3) can be sharpened for C. In this article, this question is answered positively. We obtain the weak LYM inequality and the strong LYM inequality for t-antichains which are two improvements of the standard LYM inequality (3).

There is a strong application background to study t-antichains. Actually, from refs.

[9] and [10] a t-antichain is a t-error correcting/all unidirectional error-detecting (tEC/AUED) code. In ref. [11], the weak LYM inequality and the strong LYM inequality derived in this article have been applied to systematic tEC/AUED codes to obtain new lower bounds on the redundancies of the codes. For more about tEC/AUED codes, one can consult references [8—15].

This paper is organized as follows. In sec. 1, we derive the weak LYM inequality. In sec. 2, we derive the strong LYM inequality. In sec. 3, we give detailed proofs of the main lemmas for the main results.

1 Weak LYM inequalities for t-antichains

Let C be a t-antichain in $P_n=\{0, 1\}^n$. Define

$$\mathscr{E}_i(\boldsymbol{c}) \triangleq \{\boldsymbol{x}\in P_n: |\boldsymbol{x}|=|\boldsymbol{c}|+t-2i \text{ and } d_H(\boldsymbol{x}, \boldsymbol{c})\leqslant t\} \tag{9}$$

for $i=0, 1, 2,\cdots, t$ and $\boldsymbol{c}\in C$.

For example, if $\boldsymbol{c}=111000$ and $\boldsymbol{t}=2$, then

$$\mathscr{E}_0(111000)=\{111110,\ 111011,\ 111101\},\ \mathscr{E}_2(111000)=\{001000,\ 010000,\ 100000\},$$

$$\mathscr{E}_1(111000)=\{111000,011100,011010,011001,101100,101010,101001,110100,110010,110001\}.$$

Clearly, from the definition of $\mathscr{E}_i(\boldsymbol{c})$ and Theorem 1, we have

Lemma 1. *If C is a t-antichain, then for each i with $0\leqslant i\leqslant t$ and each $\boldsymbol{c}\in C$,*

$$|\mathscr{E}_i(\boldsymbol{c})|=\sum_{j=0}^{i}\binom{|\boldsymbol{c}|}{j}\binom{n-|\boldsymbol{c}|}{t-2i+j}, \tag{10}$$

$\mathscr{E}_i(\boldsymbol{c})\cap\mathscr{E}_i(\boldsymbol{c}')=\varnothing$ when $\boldsymbol{c}\neq\boldsymbol{c}'$, and $\mathscr{E}_i(C)\triangleq\bigcup_{c\in C}\mathscr{E}_i(\boldsymbol{c})$ is an antichain.

Define

$$\mathscr{M}_n(m,\ t,\ i)\triangleq\sum_{j=0}^{i}\binom{m+t-2i}{t-2i+j}\binom{n-m-t+2i}{j} \tag{11}$$

for $i=0, 1, 2,\cdots, t$ and $m=0, 1, 2,\cdots$. We have the following result.

Theorem 1 (Weak LYM inequality for t-antichains). *If C is a t-antichain, then for each i with $0\leqslant i\leqslant t$,*

$$\sum_{c\in C}\frac{\mathscr{M}_n(|\boldsymbol{c}|,\ t,\ i)}{\binom{n}{|\boldsymbol{c}|}}\leqslant 1. \tag{12}$$

Proof. By Lemma 1, $\mathscr{E}_i(C)$ is an antichain. From (9) and the LYM inequality (3),

$$\sum_{c\in C}\frac{|\mathscr{E}_i(\boldsymbol{c})|}{\binom{n}{|\boldsymbol{c}|+t-2i}}\leqslant 1.$$

By (10),

$$\sum_{c\in C}\frac{\sum\limits_{j=0}^{i}\binom{|c|}{j}\binom{n-|c|}{t-2i+j}}{\binom{n}{|c|+t-2i}}=\sum_{c\in C}\frac{\sum\limits_{j=0}^{i}\binom{|c|+t-2i}{t-2i+j}\binom{n-|c|-t+2i}{j}}{\binom{n}{|c|}}\leqslant 1. \tag{13}$$

Combining (11) and (13) gives (12). Q.E.D.

Since the t-antichain with the n-vectors being the binary complements of the ones in a t-antichain is also a t-antichain, Theorem 1 implies the following corollary.

Corollary 1. *If C is a t-antichain, then for $i=0, 1,\cdots, t$,*

$$\sum_{c\in C}\frac{\mathscr{M}_n(n-|c|,\ t,\ i)}{\binom{n}{n-|c|}}\leqslant 1. \tag{14}$$

We now present a property of the coefficients in the weak LYM inequality. From (11) and (12), we can see that $\mathscr{M}_n(|c|, t, i)$ depends on $|c|$, t and i. But when we add them together for i from 0 to t, the summation turns out to be independent of the n-vector weight $|c|$. This can be stated as the following theorem.

Theorem 2. *The following summation,*

$$\sum_{i=0}^{t}\mathscr{M}_n(m,\ t,\ i)=\sum_{i=0}^{t}\sum_{j=0}^{i}\binom{m+t-2i}{t-2i+j}\binom{n-m-t+2i}{j}\triangleq S(n,\ t), \tag{15}$$

is independent of m when $n-t\geqslant m\geqslant t$.

Note. From (8), $n-t\geqslant m\geqslant t$ is satisfied when m is the weight of an n-vector in a non-trivial t-antichain and $S(n, t)>t+1$. The proof of this theorem can be found in section 3.

Corresponding to the basic LYM inequality (3) for antichains we have a stronger result for t-antichains as follows.

Corollary 2. *If C is a non-trivial t-antichain, then*

$$\sum_{c\in C}\frac{1}{\binom{n}{|c|}}\leqslant\frac{t+1}{S(n,\ t)}<1, \tag{16}$$

where $S(n, t)$ is defined by (15).

2 Strong LYM inequality for *t*-antichains

In this section, we improve the inequality (12) by counting the maximal chains passing through $\mathscr{E}_i(C)$ more carefully. The weak LYM-inequality is used in the proof of the following Lemma 3, which plays a key role in the proof of the following strong LYM inequality.

For each $i \in \{1, 2, \cdots, t\}$, define

$$\overline{\mathscr{M}}_n(m, t, 0) = \mathscr{M}_n(m, t, 0),$$

$$\overline{\mathscr{M}}_n(m, t, i) = \max\{\mathscr{M}_n(m, t, i-1), \mathscr{M}_n(m, t, i)\}, \tag{17}$$

where m is an arbitrary nonnegative integer.

Theorem 3 (Strong LYM inequality for t-antichains). *If C is a t-antichain, then for each i with $0 \leqslant i \leqslant t$,*

$$\sum_{c \in C} \frac{\overline{\mathscr{M}}_n(|c|, t, i)}{\binom{n}{|c|}} \leqslant 1. \tag{18}$$

Similar to Corollary 1, we have the following Corollary 3.

Corollary 3. *If C is a t-antichain, then for each i with $0 \leqslant i \leqslant t$,*

$$\sum_{c \in C} \frac{\overline{\mathscr{M}}_n(n-|c|, t, i)}{\binom{n}{n-|c|}} \leqslant 1. \tag{19}$$

Proof of Theorem 3. For $i=0$, (18) is just (12). So we only need to prove (18) for $i=1, 2, \cdots, t$. To do so, for each i with $1 \leqslant i \leqslant t$ and each $\boldsymbol{c} \in C$, define

$$A_i(\boldsymbol{c}) = \{X : X \text{ is a maximal chain in } P_n \text{ and } \exists\, \boldsymbol{x} \in \mathscr{E}_i(\boldsymbol{c}) \text{ such that } X \text{ passes } \boldsymbol{x}\}. \tag{20}$$

Since $\mathscr{E}_i(c)$ is an antichain for each $i \in \{1, 2, \cdots, t\}$, from Lemma 1 and (9)

$$|A_i(\boldsymbol{c})| = |\mathscr{E}_i(\boldsymbol{c})|(|\boldsymbol{c}| + t - 2i)!\,(n - |\boldsymbol{c}| - t + 2i)! \tag{21}$$

for $i=1, 2, \cdots, t$. Moreover,

$$\left|\bigcup_{c \in C} A_i(\boldsymbol{c})\right| = \sum_{c \in C} |A_i(\boldsymbol{c})|. \tag{22}$$

Because $\mathscr{E}_i(C)$ is also an antichain for each i by Lemma 1, (22) follows from the fact that $A_i(\boldsymbol{c})$ are disjoint for different $\boldsymbol{c} \in C$. In the following we count all maximal chains in

$$\left(\bigcup_{c \in C} A_{i-1}(\boldsymbol{c})\right) \cup \left(\bigcup_{c \in C} A_i(\boldsymbol{c})\right)$$

for $i=1, 2, \cdots, t$. Since $\mathscr{E}_{i-1}(C) \cup \mathscr{E}_i(C)$ is no longer an antichain, generally

$$\left|\left(\bigcup_{c \in C} A_{i-1}(\boldsymbol{c})\right) \cup \left(\bigcup_{c \in C} A_i(\boldsymbol{c})\right)\right| \neq \left|\bigcup_{c \in C} A_{i-1}(\boldsymbol{c})\right| + \left|\bigcup_{c \in C} A_i(\boldsymbol{c})\right|.$$

For $i=1, 2, \cdots, t$, by defining

$$B_i(\boldsymbol{c}) \triangleq A_{i-1}(\boldsymbol{c}) \cup \{ A_i(\boldsymbol{c}) - \bigcup_{\boldsymbol{c}' \in C} A_{i-1}(\boldsymbol{c}') \}, \tag{23}$$

we obtain

$$\left| \bigcup_{\boldsymbol{c} \in C} B_i(\boldsymbol{c}) \right| = \sum_{\boldsymbol{c} \in C} |\boldsymbol{B}_i(\boldsymbol{c})|, \tag{24}$$

which follows from the fact that $B_i(\boldsymbol{c})$ are disjoint for different $\boldsymbol{c} \in C$ and fixed i. Therefore,

$$\sum_{\boldsymbol{c} \in C} |B_i(\boldsymbol{c})| \leqslant n!\,. \tag{25}$$

On the other hand, for each $i \in \{1, 2, \cdots, t\}$,

$$\begin{aligned} \sum_{\boldsymbol{c} \in C} |B_i(\boldsymbol{c})| &= \sum_{\boldsymbol{c} \in C} \left\{ |A_{i-1}(\boldsymbol{c})| + \left| A_i(\boldsymbol{c}) - \bigcup_{\boldsymbol{c}' \in C} A_{i-1}(\boldsymbol{c}') \right| \right\} \\ &= \sum_{\boldsymbol{c} \in C} \left\{ |A_{i-1}(\boldsymbol{c})| + |A_i(\boldsymbol{c})| - \left| \bigcup_{\boldsymbol{c}' \in C} (A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c}')) \right| \right\} \\ &\overset{\text{by (22)}}{=} \sum_{\boldsymbol{c} \in C} \left\{ |A_{i-1}(\boldsymbol{c})| + |A_i(\boldsymbol{c})| - |A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c})| - \sum_{\boldsymbol{c}' \in C,\ \boldsymbol{c}' \neq \boldsymbol{c}} |A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c}')| \right\}. \end{aligned} \tag{26}$$

To estimate

$$|A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c})| \text{ and } \sum_{\boldsymbol{c}' \in C,\ \boldsymbol{c}' \neq \boldsymbol{c}} |A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c}')|,$$

we need the following two lemmas to be proved in section 3.

Lemma 2. *For* $i = 1, 2, \cdots, t$ *and* $\boldsymbol{c} \in C$,

$$|A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c})| = |A_{i-1}(\boldsymbol{c})| - n!\, \frac{\binom{|\boldsymbol{c}| + t - 2i}{t - i + 1} \binom{n - |\boldsymbol{c}| - t + 2i - 2}{i - 1}}{\binom{n}{|\boldsymbol{c}|}}. \tag{27}$$

Lemma 3. *For* $i = 1, 2, \cdots, t$ *and* $\boldsymbol{c} \in C$,

$$\sum_{\boldsymbol{c}' \in C,\ \boldsymbol{c}' \neq \boldsymbol{c}} |A_i(\boldsymbol{c}) \cap A_{i-1}(\boldsymbol{c}')| \leqslant n!\, \frac{\binom{|\boldsymbol{c}| + t - 2i}{t - i + 1} \binom{n - |\boldsymbol{c}| - t + 2i - 2}{i - 1}}{\binom{n}{|\boldsymbol{c}|}}. \tag{28}$$

By (21) and (10),

$$\begin{aligned} |A_i(\boldsymbol{c})| &= |\mathscr{E}_i(\boldsymbol{c})| (|\boldsymbol{c}| + t - 2i)!\, (n - |\boldsymbol{c}| - t + 2i)! \\ &= \sum_{j=0}^{i} \binom{|\boldsymbol{c}|}{j} \binom{n - |\boldsymbol{c}|}{t - 2i + j} (|\boldsymbol{c}| + t - 2i)!\, (n - |\boldsymbol{c}| - t + 2i)! \end{aligned}$$

$$=n!\,\frac{\sum_{j=0}^{i}\binom{m+t-2i}{t-2i+j}\binom{n-m-t+2i}{j}}{\binom{n}{|c|}}=n!\,\frac{\mathscr{M}_n(|c|,\ t,\ i)}{\binom{n}{|c|}}. \tag{29}$$

From (29), we can see that to prove the theorem we need to prove only

$$|B_i(c)|\geqslant\max\{|A_i(c)|,\ |A_{i-1}(c)|\}. \tag{30}$$

While the inequality $|B_i(c)|\geqslant|A_{i-1}(c)|$ is obvious from the definition of $B_i(c)$, the inequality $|B_i(c)|\geqslant|A_i(c)|$ is derived as follows:

$$\begin{aligned}|B_i(c)|&=|A_{i-1}(c)|+|A_i(c)|-|A_i(c)\cap A_{i-1}(c)|-\sum_{c'\in C,\,c'\neq c}|A_i(c)\cap A_{i-1}(c')|\\ &\overset{\text{by (27)}}{=}|A_{i-1}(c)|+|A_i(c)|-|A_{i-1}(c)|+n!\,\frac{\binom{|c|+t-2i}{t-i+1}\binom{n-|c|-t+2i-2}{i-1}}{\binom{n}{|c|}}\\ &\quad-\sum_{c'\in C,\,c'\neq c}|A_i(c)\cap A_{i-1}(c)|\overset{\text{by (28)}}{\geqslant}|A_i(c)|.\end{aligned} \tag{31}$$

Therefore by (29) and (30) and the definition of $\overline{\mathscr{M}}_n(m,\ t,\ i)$,

$$\sum_{c\in C}\frac{\overline{\mathscr{M}}_n(|c|,\ t,\ i)}{\binom{n}{|c|}}\leqslant 1.$$

This proves Theorem 3. Q.E.D.

3 Proofs of Theorem 2

In the proof of Theorem 2, we need a lemma.

Lemma 4. *For any fixed non-negative integers a and t, the following summation*

$$\sum_{i=0}^{t}\binom{n-m+ai}{i}\binom{m-ai}{t-i} \tag{32}$$

is independent of m for $n\geqslant m\geqslant at$.

Proof. When $a=0$, (32) is equal to $\binom{n}{t}$ for $n\geqslant m$. So we only need to prove Lemma 4 for $a>0$. Let

$$f_a(n,\ m,\ t)\overset{\Delta}{=}\sum_{i=0}^{t}\binom{n-m+ai}{i}\binom{m-ai}{t-i}.$$

Fixing a, we use induction on t.

When $t=0$, $f_a(n, m, t)=1$ for $n\geqslant m$. So $f_a(n, m, t)$ is independent of m when $n\geqslant m\geqslant at$.

Assume that $f_a(n, m, t')$ is independent of m when $n\geqslant m\geqslant at'$ for $0\leqslant t'\leqslant t-1$; that is, there exists a function $g_a(n, t-j)$ of n and $t-j$ such that

$$f_a(n, m, t-j)=g_a(n, t-j), \tag{33}$$

where $n\geqslant m\geqslant a(t-j)$ and $1\leqslant j\leqslant t$. We now prove that $f_a(n, m, t)$ is also independent of m for $n\geqslant m\geqslant at$. To do so, we use induction on n.

First, when $n=at$, this is clear because there is only one possible m which is at. For $n=at+1$, it is enough to prove that

$$f_a(at+1, at, t)=f_a(at+1, at+1, t). \tag{34}$$

In fact,

$$f_a(at+1, at, t)=\sum_{i=0}^{t}\binom{1+ai}{i}\binom{at-ai}{t-i}=\sum_{i=1}^{t}\binom{1+a(t-i)}{t-i}\binom{ai}{i}=f_a(at+1, at+1, t).$$

Assume that $f_a(n', m, t)$ is independent of m when $n-1\geqslant n'\geqslant m\geqslant at$; that is, there exists a function $h_a(n', t)$ of n' and t such that

$$f_a(n', m, t)=h_a(n', t), \tag{35}$$

where $n-1\geqslant n'\geqslant m\geqslant at$. We prove that $f_a(n, m, t)$ is also independent of m when $n\geqslant m\geqslant at$.

In the case of $n=m=at$ and $n=at+1$, this has been proved. So we assume $n>at+1$.

(i) When $n-2\geqslant m\geqslant at$,

$$\begin{aligned}f_a(n, m, t)&=\sum_{i=0}^{t}\binom{n-m+ai}{i}\binom{m-ai}{t-i}=\sum_{i=0}^{t}\binom{n-1-m+ai}{i}\binom{m-ai}{t-i}\\&\quad+\sum_{i=1}^{t}\binom{n-m+a-1+a(i-1)}{i-1}\binom{m-a-a(i-1)}{t-1-(i-1)}\\&=f_a(n-1, m, t)+f_a(n-1, m-a, t-1).\end{aligned} \tag{36}$$

By (33) and (35),

$$f_a(n, m, t)=h_a(n-1, t)+g_a(n-1, t-1). \tag{37}$$

(ii) When $n\geqslant m\geqslant at+2$,

$$\sum_{i=0}^{t}\binom{n-m+ai}{i}\binom{m-ai}{t-i}=\sum_{i=0}^{t}\binom{n-m+ai}{i}\binom{m-ai-1}{t-i}$$

$$+\sum_{i=0}^{t-1}\binom{n-m+ai}{i}\binom{m-ai-1}{t-i-1}=f_a(n-1,\ m-1,\ t)+f_a(n-1,\ m-1,\ t-1). \tag{38}$$

By (33) and (35),

$$f_a(n,\ m,\ t)=h_a(n-1,\ t)+g_a(n-1,\ t-1). \tag{39}$$

Combining (37) and (39), $f_a(n,\ m,\ t)=h_a(n-1,\ t)+g_a(n-1,\ t-1)$ for $n\geqslant m\geqslant at$. This proves that $f_a(n,\ m,\ t)$ is independent of m when $n\geqslant m\geqslant at$. By induction, Lemma 4 is proved. Q.E.D.

Moreover, from (36) and (38),

$$f_a(n,\ m,\ t)=f_a(n,\ t)=f_a(n-1,\ t)+f_a(n-1,\ t-1). \tag{40}$$

This is a recursive formula for the summation in (32).

Proof of Theorem 2.

Let

$$J(n,\ m,\ t)\triangleq\sum_{i=0}^{t}\sum_{j=1}^{i}\binom{m+t-2i}{t-2i+j}\binom{n-m-t+2i}{j}.$$

We use induction on t.

When $t=0$, $J(n,\ m,\ t)=1$ for all $n\geqslant m\geqslant 0$. So $J(m,\ n,\ t)$ is independent of m when $n\geqslant m\geqslant 0$.

When $t=1$, for $n-1\geqslant m\geqslant 1$

$$J(n,\ m,\ t)=\binom{m+1}{1}+\binom{n-m+1}{1}=n+2.$$

So $J(n,\ m,\ t)$ is independent of m when $n-1\geqslant m\geqslant 1$.

Assume that $J(n,\ m,\ t)$ is independent of m when $n-t\geqslant m\geqslant t$. We prove that $J(n,\ m,\ t+2)$ is also independent of m when $n-(t+2)\geqslant m\geqslant t+2$.

Let $i'=t-2i$. Then,

$$J(n,\ m,\ t+2)=\sum_{i'\in\{t+2,\ t,\ t-2,\cdots,\ -t,\ -t-2\}}\ \sum_{j=0}^{\frac{t-i'}{2}+1}\binom{m+i'}{i'+j}\binom{n-m-i'}{j}$$

$$=\sum_{i'\in\{t+2,\ -t-2\}}\ \sum_{j=0}^{\frac{t-i'}{2}}\binom{m+i'}{i'+j}\binom{n-m-i'}{j}+\sum_{i'\in\{t,\ t-2,\cdots,\ -t+2,\ -t\}}\ \sum_{j=0}^{\frac{t-i'}{2}}\binom{m+i'}{i'+j}\binom{n-m-i'}{j}$$

$$+\sum_{i'\in\{t+2,\ t,\ t-2,\cdots,\ -t,\ -t-2\}}\binom{m+i'}{i'+\frac{t-i'}{2}+1}\binom{n-m-i'}{\frac{t-i'}{2}+1}$$

$$=0+J(n,\ m,\ t)+\sum_{i=0}^{t+2}\binom{m+t-2i+2}{t+2-i}\binom{n-m-t+2i-2}{i}.$$

When $n-(t+2)\geqslant m\geqslant t+2$, we have $n-t\geqslant m\geqslant t$ and $n\geqslant m+t+2\geqslant 2(t+2)$. Therefore, by the hypothesis of the induction and Lemma 4 with $a:=2$, $m:=m+t+2$, $J(n,\ m,\ t)$ and

$$\sum_{i=0}^{t+2}\binom{m+t-2i+2}{t+2-i}\binom{n-m-t+2i-2}{i}$$

are independent of m when $n-(t+2)\geqslant m\geqslant t+2$. So $J(n,\ m,\ t+2)$ is independent of m when $n-(t+2)\geqslant m\geqslant t+2$. By induction, Theorem 2 is proved. Q.E.D.

In the proofs of Lemma 2 and Lemma 3, we make the following assumptions.

A maximal chain X always means $X=\{\boldsymbol{a}_0,\ \boldsymbol{a}_1,\cdots,\ \boldsymbol{a}_n\}$ with $\boldsymbol{a}_0\leqslant\boldsymbol{a}_1\leqslant\cdots\leqslant\boldsymbol{a}_n$ such that $\boldsymbol{a}_i\in P_n$ and $|\boldsymbol{a}_i|=i$ for $i=0,\ 1,\cdots,\ n$.

Without loss of generality, to prove Lemma 2 and Lemma 3 for an n-vector $\boldsymbol{c}$ in C, we only need to prove them for $\boldsymbol{c}=\underbrace{11\cdots1}_{|\boldsymbol{c}|}\ \underbrace{00\cdots0}_{n-|\boldsymbol{c}|}$ with $c_1=c_2=\cdots=c_{|\boldsymbol{c}|}=1$ and $c_{|\boldsymbol{c}|+1}=c_{|\boldsymbol{c}|+2}=\cdots=c_n=0$.

Proof of Lemma 2.

$$|A_i(\boldsymbol{c})\cap A_{i-1}(\boldsymbol{c})|=\sum_{\boldsymbol{x}\in\mathscr{E}_{i-1}(\boldsymbol{c})}|\{X{:}X\in A_i(\boldsymbol{c})\text{ and }X\text{ passes }\boldsymbol{x}\}|$$

$$\overset{1}{=}|A_{i-1}(\boldsymbol{c})|-\sum_{\boldsymbol{x}\in\mathscr{E}_{i-1}(\boldsymbol{c})}|\{X{:}X\text{ is a maximal chain in }P_n,\ X\notin A_i(\boldsymbol{c})\text{ and }X\text{ passes }\boldsymbol{x}\}|,\tag{41}$$

where step 1 is because $\mathscr{E}_{i-1}(\boldsymbol{c})$ is an antichain by Lemma 1. Let us check what is

$$D(\boldsymbol{x})\overset{\Delta}{=}|\{X{:}X\text{ is a maximal chain in }P_n,\ X\notin A_i(\boldsymbol{c})\text{ and }X\text{ passes }\boldsymbol{x}\}|$$

for $\boldsymbol{x}\in\mathscr{E}_{i-1}(\boldsymbol{c})$.

If a maximal chain X passes $\boldsymbol{x}\in\mathscr{E}_{i-1}(\boldsymbol{c})$, then by (9),

$$\boldsymbol{a}_{|\boldsymbol{c}|+t-2i+2}=\boldsymbol{x}.\tag{42}$$

Claim 1. *If* $N(\boldsymbol{x},\ \boldsymbol{c})<t-i+1$, *then* $D(\boldsymbol{x})=0$.

To prove Claim 1, let X be an arbitrary maximal chain in P_n such that X passes $\boldsymbol{x}$. Thus (42) is true. We now prove $X\in A_i(\boldsymbol{c})$. It is enough to show that $\boldsymbol{y}=\boldsymbol{a}_{|\boldsymbol{c}|+t-2i}\in\mathscr{E}_i(\boldsymbol{c})$. To prove this, by (9) and $|\boldsymbol{y}|=|\boldsymbol{c}|+t-2i$ we only need to check if $d_H(\boldsymbol{y},\ \boldsymbol{c})\leqslant t$. By (9) and (42),

$$N(\boldsymbol{c},\ \boldsymbol{x})=N(\boldsymbol{x},\ \boldsymbol{c})+|\boldsymbol{c}|-|\boldsymbol{x}|=N(\boldsymbol{x},\ \boldsymbol{c})+|\boldsymbol{c}|-|\boldsymbol{c}|+2i-2-t=N(\boldsymbol{x},\ \boldsymbol{c})+2i-2-t.\tag{43}$$

By $N(\boldsymbol{x},\ \boldsymbol{c})\leqslant t-i$,

$$d_H(\boldsymbol{x},\ \boldsymbol{c})\leqslant 2N(\boldsymbol{x},\ \boldsymbol{c})+2i-2-t\leqslant t-2.\tag{44}$$

Since $N(\boldsymbol{x}, \boldsymbol{y})=2$ and $N(\boldsymbol{y}, \boldsymbol{x})=0$, we have $N(\boldsymbol{c}, \boldsymbol{y})=N(\boldsymbol{c}, \boldsymbol{x})+N(\boldsymbol{x}, \boldsymbol{y})=N(\boldsymbol{c}, \boldsymbol{x})+2$, and $N(\boldsymbol{y}, \boldsymbol{c})\leqslant N(\boldsymbol{y}, \boldsymbol{x})+N(\boldsymbol{x}, \boldsymbol{c})=N(\boldsymbol{x}, \boldsymbol{c})$. By (44), we have $d_H(\boldsymbol{y}, \boldsymbol{c})=N(\boldsymbol{y}, \boldsymbol{c})+N(\boldsymbol{c}, \boldsymbol{y})\leqslant d_H(\boldsymbol{x}, \boldsymbol{c})+2\leqslant t$. Therefore, any maximal chain X passing $\boldsymbol{x}\in\mathscr{E}_{i-1}(\boldsymbol{c})$ passes a vector $\boldsymbol{y}\in\mathscr{E}_i(\boldsymbol{c})$. This proves Claim 1 by the definition of $A_i(\boldsymbol{c})$. Q.E.D.

Claim 2. *If* $N(\boldsymbol{x}, \boldsymbol{c})=t-i+1$, *then*

$$D(\boldsymbol{x})=\binom{|\boldsymbol{c}|-i+1}{2}2!\,(|\boldsymbol{c}|+t-2i)!\,(n-|\boldsymbol{c}|-t+2i-2)!\,. \tag{45}$$

For simplicity, we assume that

$$\boldsymbol{x}=\underbrace{00\cdots0}_{i-1}\,\overbrace{11\cdots1}^{|\boldsymbol{c}|-i+1}\,\overbrace{11\cdots1}^{t-i+1}\,\underbrace{00\cdots0}_{n-|\boldsymbol{c}|-t+i-1}\quad;$$

that is, $x_1=x_2=\cdots=x_{i-1}=0$, $x_i=x_{i+1}=\cdots=x_{|\boldsymbol{c}|}=x_{|\boldsymbol{c}|+1}=\cdots=x_{|\boldsymbol{c}|+t-i+1}=1$, and $x_{|\boldsymbol{c}|+t-i+2}=x_{|\boldsymbol{c}|+t-i+3}=\cdots=x_n=0$. For other $\boldsymbol{x}$, the proof is the same.

Let X be an arbitrary maximal chain in P_n passing $\boldsymbol{x}$ such that $\boldsymbol{a}_l=\boldsymbol{x}$, where $l=|\boldsymbol{x}|+t-2i+2$ and $X=\{\boldsymbol{a}_0, \boldsymbol{a}_1,\cdots, \boldsymbol{a}_n\}$ with $\boldsymbol{a}_j=(a_{j,1}, a_{j,2},\cdots, a_{j,n})$ for $j=1, 2,\cdots, n$.

If $a_{l-1,m}=0$ or $a_{l-2,m}=0$ for some $m\in\{|\boldsymbol{c}|+1, |\boldsymbol{c}|+2,\cdots, |\boldsymbol{c}|+t-i+1\}$, then $\boldsymbol{a}_{l-2}\in\mathscr{E}_i(\boldsymbol{c})$ by (9) because

$$d_H(\boldsymbol{a}_{l-2}, \boldsymbol{c})\leqslant d_H(\boldsymbol{a}_l, \boldsymbol{c})-1+1\leqslant t.$$

If $a_{l-1,m_1}=0$ for an $m_1\in\{i, i+1,\cdots, |\boldsymbol{c}|\}$ and $a_{l-2,m_2}=0$ for an $m_2\in\{i, i+1,\cdots, |\boldsymbol{c}|\}$ different from m_1, then $\boldsymbol{a}_j\notin\mathscr{E}_i(\boldsymbol{c})$ for any $j\in\{0, 1, 2,\cdots, n\}$. To prove this, we only need to check if $\boldsymbol{a}_{l-2}\in\mathscr{E}_i(\boldsymbol{c})$ because the weight of any vector in $\mathscr{E}_i(\boldsymbol{c})$ is $l-2$ by (9). Since $a_{l-1,m_1}=a_{l-2,m_2}=0$ for two different $m_1, m_2\in\{i, i+1,\cdots, |\boldsymbol{c}|\}$,

$$N(\boldsymbol{c}, \boldsymbol{a}_{l-2})=i-1+2=i+1 \text{ and } N(\boldsymbol{a}_{l-2}, \boldsymbol{c})=N(\boldsymbol{a}_l, \boldsymbol{c})=t-i+1.$$

Therefore, $d_H(\boldsymbol{c}, \boldsymbol{a}_{l-2})=i+1+t-i+1=t+2>t$. This proves that $\boldsymbol{a}_{l-2}\notin\mathscr{E}_i(\boldsymbol{c})$ by (9).

Combining the above two cases for $\boldsymbol{a}_{l-1}$ and $\boldsymbol{a}_{l-2}$, we see $\boldsymbol{a}_{l-2}\notin\mathscr{E}_i(\boldsymbol{c})$ if and only if $a_{l-1,m_1}=a_{l-2,m_2}=0$ for two different m_1 and m_2 in $\{i, i+1,\cdots, |\boldsymbol{c}|\}$. We have

$$\binom{|\boldsymbol{c}|-i+1}{2}$$

different pairs $\{m_1, m_2\}\subset\{i, i+1,\cdots, |\boldsymbol{c}|\}$. Moreover, for different order of m_1 and m_2, $\boldsymbol{a}_{l-1}$ is different. Therefore, there are

$$\binom{|\boldsymbol{c}|-i+1}{2}2!$$

different arrangements for $\boldsymbol{a}_{l-1}$ and $\boldsymbol{a}_{l-2}$.

Other $\boldsymbol{a}_j$'s with $j\notin\{l-2, l-1, l\}$ can be selected arbitrarily to make $\{\boldsymbol{a}_0, \boldsymbol{a}_1,\cdots, \boldsymbol{a}_n\}$ a maximal chain in P_n. There are $(n-|c|-t+2i-2)!$ different arrangements for $\{a_{l+1}, a_{l+2}, \cdots, a_n\}$, and $(|c|+t-2i)!$ different arrangements for $\{a_0, a_1, \cdots, a_{l-3}\}$. Therefore, the total number of maximal chains passing $x \in \mathscr{E}_{i-1}(c)$ without passing any $y \in \mathscr{E}_i(c)$ is

$$\binom{|c|-i+1}{2} 2!\,(|c|+t-2i)!\,(n-|c|-t+2i-2)!\,.$$

This proves Claim 2. Q.E.D.

Noticing that for $x \in \mathscr{E}_{i-1}(c)$ there are only the above two cases in Claim 1 and Claim 2 for $N(x, c)$, we have

$$\begin{aligned}\sum_{x\in\mathscr{E}_{i-1}(c)} D(x) &= \sum_{x\in\mathscr{E}_{i-1}(c),\ N(x,\ c)=t-i+1} \binom{|c|-i+1}{2} 2!\,(|c|+t-2i)!\,(n-|c|-t+2i-2)! \\ &\overset{1}{=} \binom{|c|}{i-1}\binom{n-|c|}{t-i+1}\binom{|c|-i+1}{2} 2!\,(|c|+t-2i)!\,(n-|c|-t+2i-2)! \\ &= n!\,\frac{\binom{n-|c|-t+2i-2}{i-1}\binom{|c|+t-2i}{t-i+1}}{\binom{n}{|c|}},\end{aligned} \tag{46}$$

where step 1 is because there are

$$\binom{|c|}{i-1}\binom{n-|c|}{t-i+1}$$

different x such that $x \in \mathscr{E}_{i-1}(c)$ and $N(x, c)=t-i+1$. Lemma 2 is implied by (41) and (46). Q.E.D.

Proof of Lemma 3.

Let

$$E(c) \overset{\Delta}{=} \bigcup_{c'\in C,\ c'\neq c} A_i(c) \cap A_{i-1}(c'),$$

and

$$C(c) \overset{\Delta}{=} \{c' : c' \in C,\ c' \neq c \text{ and } A_i(c) \cap A_{i-1}(c') \neq \varnothing\}.$$

To prove Lemma 3, we first investigate some properties of $E(c)$ and $C(c)$.

Claim 3. *If $c' \in C(c)$, then $N(c, c')=t+1$.*

Proof. Let $X \in A_i(c) \cap A_{i-1}(c')$ pass an $x \in \mathscr{E}_i(c)$ and a $y \in \mathscr{E}_{i-1}(c')$. Then

$$t+1 \leqslant N(c, c') \leqslant N(c, x)+N(x, y)+N(y, c') \leqslant i+t-i+1+N(x, y).$$

To prove Claim 3, we only need to prove $N(x, y)=0$. Since X passes x and y, we have either $x \leqslant y$ or $y \leqslant x$. To prove $N(x, y)=0$, it is enough to exclude the case $y \leqslant x$. If

$\boldsymbol{y}\leqslant\boldsymbol{x}$, then $N(\boldsymbol{y},\ \boldsymbol{c})\leqslant N(\boldsymbol{y},\ \boldsymbol{x})+N(\boldsymbol{x},\ \boldsymbol{c})=t-i$. And $\boldsymbol{y}\in\mathscr{E}_{i-1}(\boldsymbol{c})$ implies $N(\boldsymbol{c}',\ \boldsymbol{y})\leqslant i-1$. Thus $N(\boldsymbol{c},\ \boldsymbol{c}')\leqslant t-1$. This contradicts the fact that $N(\boldsymbol{c},\ \boldsymbol{c}')\geqslant t+1$. Claim 3 is proved. Q.E.D.

For convenience, we introduce two notations.

For any $\boldsymbol{x}\in\{0,\ 1\}^n$ with $\boldsymbol{x}=(x_1,\ x_2,\cdots,\ x_n)$, define $\tilde{\boldsymbol{x}}=(x_1,\ x_2,\cdots,\ x_{|\boldsymbol{c}|})$ and $\hat{\boldsymbol{x}}=(x_{|\boldsymbol{c}|+1},\cdots,\ x_n)$. Then, $\boldsymbol{x}=(\tilde{\boldsymbol{x}},\ \hat{\boldsymbol{x}})$. From the assumption at the beginning of this section, $\tilde{\boldsymbol{c}}=(1,\ 1,\cdots,\ 1)$ and $\hat{\boldsymbol{c}}=(0,\ 0,\cdots,\ 0)$. With this notation, Claim 3 can be rephrased as

$$\forall \boldsymbol{c}'\in C(\boldsymbol{c}),\ |\tilde{\boldsymbol{c}}'|=|\boldsymbol{c}|-t-1. \tag{47}$$

Claim 4. *If* $x\in E_i(\boldsymbol{c})$ *and* $y\in E_{i-1}(\boldsymbol{c}')$ *are passed by some* $X\in A_i(\boldsymbol{c})\cap A_{i-1}(\boldsymbol{c}')$, *then* $\boldsymbol{x}\leqslant\boldsymbol{y}$, $N(\boldsymbol{c},\ \boldsymbol{x})=N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{x}})=i$, $N(\boldsymbol{y},\ \boldsymbol{c}')=N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{c}}')=t-i+1$, $|\tilde{\boldsymbol{x}}|=|\boldsymbol{c}|-i$, $\tilde{\boldsymbol{c}}'\leqslant\tilde{\boldsymbol{x}}$ *and* $\tilde{\boldsymbol{x}}=\tilde{\boldsymbol{y}}$.

The following is an example.

$$\begin{array}{cccc} \tilde{\boldsymbol{y}} & \overbrace{00\cdots0}^{i} & \overbrace{11\cdots1}^{t-i+1} & \overbrace{11\cdots1}^{|\boldsymbol{c}|-t-1} \\ \tilde{\boldsymbol{c}}' & 00\cdots0 & 00\cdots0 & 11\cdots1 \\ \tilde{\boldsymbol{c}} & 11\cdots1 & 11\cdots1 & 11\cdots1 \\ \tilde{\boldsymbol{x}} & 00\cdots0 & 11\cdots1 & 11\cdots1 \end{array}, \qquad \begin{array}{ccccc} \hat{\boldsymbol{y}} & \overbrace{11\cdots1}^{t-i} & 11\cdots1 & \overbrace{00\cdots0}^{i-1} & 00\cdots0 \\ \hat{\boldsymbol{c}}' & 11\cdots1 & 11\cdots1 & 11\cdots1 & 00\cdots0 \\ \hat{\boldsymbol{c}} & 00\cdots0 & 00\cdots0 & 00\cdots0 & 00\cdots0 \\ \hat{\boldsymbol{x}} & 11\cdots1 & 00\cdots0 & 00\cdots0 & 00\cdots0 \end{array}.$$

Proof. $\boldsymbol{x}\leqslant\boldsymbol{y}$ has already been proved in the proof of Claim 3. We now prove the rest of Claim 4.

Since $N(\boldsymbol{c},\ \boldsymbol{x})\leqslant i$, $N(\boldsymbol{y},\ \boldsymbol{c}')\leqslant t-i+1$ and $N(\boldsymbol{c},\ \boldsymbol{c}')=t+1$, we have

$$N(\boldsymbol{c},\ \boldsymbol{x})=N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{x}})=i \text{ and } N(\boldsymbol{y},\ \boldsymbol{c}')=t-i+1.$$

By $N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{c}}')\leqslant t-i+1$, $\tilde{\boldsymbol{x}}\leqslant\tilde{\boldsymbol{y}}$ and $N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{c}}')=t+1$, we have $N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{c}}')=t-i+1$. It is clear that $|\tilde{\boldsymbol{x}}|=|\boldsymbol{c}|-i$. Next we prove $\tilde{\boldsymbol{c}}'\leqslant\tilde{\boldsymbol{x}}$. If it is not true, that is, $N(\tilde{\boldsymbol{c}}',\ \tilde{\boldsymbol{x}})>0$, then

$$N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{c}}')=N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{x}})+N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{c}}')-N(\tilde{\boldsymbol{c}}',\ \tilde{\boldsymbol{x}})<t+1.$$

This contradicts Claim 3. To prove $\tilde{\boldsymbol{x}}=\tilde{\boldsymbol{y}}$, we see that if $\tilde{\boldsymbol{x}}\neq\tilde{\boldsymbol{y}}$, then $N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{x}})>0$. Let $J\triangleq\{j:\tilde{y}_j=1\wedge\tilde{x}_j=0\}$. Then, by $\tilde{\boldsymbol{c}}'\leqslant\tilde{\boldsymbol{x}}$, $\forall j\in J$, $\tilde{c}_j'=0$. Therefore, $N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{c}}')=N(\tilde{\boldsymbol{c}},\ \tilde{\boldsymbol{x}})+N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{c}}')-N(\tilde{\boldsymbol{y}},\ \tilde{\boldsymbol{x}})<t+1$. This also contradicts Claim 3. Claim 4 is proved. Q.E.D.

From Claim 4 we have the following consequences.

Let $J_{\tilde{x}}\triangleq\{j:\tilde{x}_j=1\}$. Then, from Claim 4, under the same assumptions, $N(\tilde{\boldsymbol{x}},\ \tilde{\boldsymbol{c}}')=t-i+1$; that is,

$$|\{j\in J_{\tilde{x}}:\tilde{c}_j'=0\}|=t-i+1. \tag{48}$$

For any $\boldsymbol{u}\in\{0,\ 1\}^n$ and $j\in\{0,\ 1,\ 2,\cdots,\ t\}$, define

$$E_j(\boldsymbol{u})\triangleq\{\boldsymbol{v}\in\mathscr{E}_j(\boldsymbol{u}):d_H(\boldsymbol{u},\ \boldsymbol{v})=t\},$$

(see (9) for the definition of $\mathscr{E}_i(\boldsymbol{u})$) and

$$A_j'(\boldsymbol{u})\triangleq\{X\in A_i(\boldsymbol{u}):X \text{ passes some } \boldsymbol{v}\in E_j(\boldsymbol{u})\},$$

(compare this with (20)). Then Claim 4 says that for any $c' \in C(c)$,

$$A_i(c) \cap A_{i-1}(c') = A_i'(c) \cap A_{i-1}'(c'). \tag{49}$$

For $x \leqslant y$, define

$$A(x, y) \triangleq \{\text{all maximal chains passing } x \text{ and } y\}.$$

Then for any two different pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ with $x_m \in E_i(c)$ and $y_m \in E_{i-1}(c')$ for $m=1, 2$, we have

$$A(x_1, y_1) \cap A(x_2, y_2) = \varnothing. \tag{50}$$

This is because for each fixed j, $E_j(c)$ is an antichain. Moreover,

$$|A(x, y)| = |x|!\,(n-|y|)!\,(|y|-|x|)!\,. \tag{51}$$

To prove Lemma 3, we divide $E(c)$ into a union of some of its subsets. To do so, we need more notations. For any $\widetilde{c}'$ with weight $|c|-t-1$, let

$$\hat{C}(\widetilde{c}') \triangleq \{\hat{c}' : c' = (\widetilde{c}', \hat{c}') \in C(c)\}.$$

Define

$$\widetilde{C}(c, \widetilde{x}) \triangleq \{\widetilde{c}' : |\widetilde{c}'| = |c|-t-1, \text{ and } \widetilde{c}' \leqslant \widetilde{x}\},$$

$$\hat{C}(\widetilde{x}) \triangleq \bigcup_{\widetilde{c}' \in \widetilde{C}(c, \widetilde{x})} \hat{C}(\widetilde{c}'),$$

and for $j \in J_{\widetilde{x}}$,

$$\hat{C}_j(\widetilde{x}) \triangleq \bigcup_{\widetilde{c}' \in \widetilde{C}(c, \widetilde{x}),\ \widetilde{c}_j' = 0} \hat{C}(\widetilde{c}').$$

The following are two more facts needed in the proof of Lemma 3.

Claim 5. *If $\hat{c}' \in \hat{C}(\widetilde{x})$, then $|\{j : \hat{c}' \in \hat{C}_j(\widetilde{x})\}| = t-i+1$.*

The proof of this claim is straightforward and therefore omitted.

For integers $0 \leqslant j \leqslant k$, let

$$\Omega_j(k) \triangleq \{x \in \{0, 1\}^k : |x| = j\}. \tag{52}$$

Claim 6. *For each $\widetilde{x} \in \Omega_{|c|-i}(|c|)$ and each $j \in J_{\widetilde{x}}$, $\hat{C}_j(\widetilde{x})$ is an i-antichain.*

Proof. For any two different $\hat{c}_1'$ and $\hat{c}_2'$ in $\hat{C}_j(\widetilde{x})$, we need to prove that $N(\hat{c}_1', \hat{c}_2') \geqslant i+1$. Suppose $\hat{c}_m' \in \hat{C}_j(\widetilde{c}_m')$ with $\widetilde{c}_m' \in \widetilde{C}(\widetilde{c}, \widetilde{x})$ for $m=1, 2$. Then, $\widetilde{c}_{1,j}' = \widetilde{c}_{2,j}' = 0$. Also, by $N(\widetilde{x}, \widetilde{c}_m') = t-i+1$ and $\widetilde{c}_m' \leqslant \widetilde{x}$ for $m=1, 2$, we have $N(\widetilde{c}_1', \widetilde{c}_2') \leqslant t-i$. But, $N(c_1', c_2') \geqslant t+1$. Therefore, $N(\hat{c}_1', \hat{c}_2') = N(c_1', c_2') - N(\widetilde{c}_1', \widetilde{c}_2') \geqslant i+1$. This proves Claim 6. Q.E.D.

Now we are ready to prove (28). By (49),

$$E(c) = \bigcup_{c' \in C,\ c' \neq c} (A_i'(c) \cap A_{i-1}'(c')) \stackrel{1}{=} \bigcup_{x \in E_i(c)} \bigcup_{c' \in C(c)} \bigcup_{y \in E_{i-1}(c'),\ x \leqslant y} A(x, y)$$

$$\stackrel{2}{=} \bigcup_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \bigcup_{\hat{x}\in\Omega_{t-i}(n-|c|)} \bigcup_{\tilde{c}'\in C(c,\tilde{x})} \bigcup_{\hat{c}'\in\hat{C}(\tilde{c}')} \bigcup_{\hat{x}\leqslant\hat{y}\leqslant\hat{c}',\ N(\hat{c}',\hat{y})=i-1} A(\boldsymbol{x},\ \boldsymbol{y})$$

$$= \bigcup_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \bigcup_{\hat{c}'\in\hat{C}(\tilde{x})} \bigcup_{\hat{y}\leqslant\hat{c}',\ N(\hat{c}',\hat{y})=i-1} \bigcup_{\hat{x}\leqslant\hat{y},\ |\hat{x}|=t-i} A(\boldsymbol{x},\ \boldsymbol{y}).$$

Therefore, by Claim 5 and eq. (50)

$$|E(c)| \leqslant \sum_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \frac{1}{t-i+1} \sum_{j\in J_{\tilde{x}}} \sum_{\hat{c}'\in\hat{C}_j(\tilde{x})} \sum_{\hat{y}\leqslant\hat{c}',\ N(\hat{c}',\hat{y})=i-1} \sum_{\hat{x}\leqslant\hat{y},\ |\hat{y}|=t-i} |A(\boldsymbol{x},\ \boldsymbol{y})|$$

$$\stackrel{3}{=} \frac{1}{t-i+1} \sum_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \sum_{j\in J_{\tilde{x}}} \sum_{\hat{c}'\in\hat{C}_j(x)} \binom{|\hat{c}'|}{i-1}\binom{|\hat{c}'|-i+1}{t-i}$$

$$\cdot (|c|+t-2i)!\,(n-|c|-|\hat{c}'|+2i-1)!\,(|\hat{c}'|-t+1)!$$

$$\stackrel{4}{\leqslant} \frac{(|c|+t-2i)!}{(t-i+1)!} \sum_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \sum_{j\in J_{\tilde{x}}} \sum_{\hat{c}'\in\hat{C}_j(\tilde{x})} |\hat{c}'|!\,(n-|c|-|\hat{c}'|+i)!$$

$$\cdot \frac{(n-|c|-t+2i-2)(n-|c|-t+2i-3)\cdots(n-|c|-t+i)}{(i-1)!}$$

$$= \frac{(|c|+t-2i)!}{(t-i+1)!}\binom{n-|c|-t+2i-2}{i-1} \sum_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \sum_{j\in J_{\tilde{x}}} \left[\sum_{\hat{c}'\in\hat{C}_j(\tilde{x})} \frac{\binom{n-|c|-|\hat{c}'|+i}{i}}{\binom{n-|c|}{|\hat{c}'|}}\right]\cdot i!\,(n-|c|)!$$

$$\stackrel{5}{\leqslant} \frac{(|c|+t-2i)!\,i!\,(n-|c|)!}{(t-i+1)!}\binom{n-|c|-t+2i-2}{i-1}\cdot \sum_{\tilde{x}\in\Omega_{|c|-i}(|c|)} \sum_{j\in J_{\tilde{x}}} 1$$

$$\stackrel{6}{=} \frac{(|c|+t-2i)!\,i!\,(n-|c|)!}{(t-i+1)!}\binom{n-|c|-t+2i-2}{i-1}\binom{|c|}{i}(|c|-i)$$

$$= n!\binom{|c|+t-2i}{t-i+1}\binom{n-|c|-t+2i-2}{i-1}\frac{1}{\binom{n}{|c|}}.$$

Step 1 is from Claim 4. Step 2 is because from $E_i(c)=\{(\tilde{x},\ \hat{x}):\tilde{x}\in\Omega_{|c|-i}(|c|)$ and $N(\hat{x},\ \hat{c})=t-i\}$, and $\tilde{x}=\tilde{y}$ by Claim 4, we have $\{y\in E_{i-1}(c'):x\leqslant y\}=\{(\tilde{x},\ \hat{y}):N(\hat{c}',\ \hat{y})=i-1$ and $\hat{x}\leqslant\hat{y}\}$, where $\Omega_{|c|-i}(|c|)$ is defined by (52). Step 3 is because

$$|\{\hat{y}:\hat{y}\leqslant\hat{c}' \text{ and } N(\hat{c}',\ \hat{y})=i-1\}|=\binom{|\hat{c}'|}{i-1},$$

and

$$|\{\hat{x}:\hat{x}\leqslant\hat{y} \text{ and } |\hat{x}|=t-i\}|=\binom{|\hat{y}|}{t-i}=\binom{|\hat{c}'|-i+1}{t-i}.$$

Step 4 is because $|\hat{c}'|\geqslant t+1$ by $N(c',\ c)\geqslant t+1$. From Claim 6, $\hat{C}_j(\tilde{x})$ is an i-antichain, which implies

$$\sum_{\hat{c}' \in \hat{C}_j(\tilde{x})} \frac{\binom{n-|c|-|\hat{c}'|+i}{i}}{\binom{n-|c|}{|\hat{c}'|}} \leqslant 1$$

by Corollary 1 with i and t in (14) replaced by 0 and i respectively. This justifies step 5. Step 6 is because $|\Omega_{|c|-i}(|c|)| = \binom{|c|}{i}$ and $|J_{\tilde{x}}| = |c| - i$. The latter one is from the fact $N(\tilde{c}, \tilde{x}) = i$. This concludes the proof of Lemma 3. Q.E.D.

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