and its Fourier transform is shown in Fig. 5. Comparing with Fig. 3 ($J = 1$) and Fig. 4 ($j = 2$), we see the approximation is also good.

At last, taking $J = 0$ and applying algorithm 2 (the algorithm in [9]), we obtain the initial value ν_n^0, which corresponds to the WS coefficients of $f(t)$ in $\mathbf{V}_0$, as shown in Fig. 6. Comparing it with the above results ($J = 0$ in Fig. 3 and $j = 1$ in Fig. 4), one can see that the obvious aliasing occurred at high frequency (0.5 Hz).

V. Conclusion

In this work, the initialization from WS to DWT has been studied. We have formulated the problem and discussed methods for its solution. Two algorithms for initialization have been proposed, which are more flexible and computationally efficient. They provide significant improvement in accurate approximation of WS coefficients over the Mallat algorithm [6] and the algorithm in [9], as shown in numerical examples. A basic prerequisite for the efficiency of our algorithms is that the sampling scheme must maintain the information of signal as much as possible. How to choose the best multiresolution subspace for initialization is an open problem in our algorithms.

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On Characterization of the Optimal Biorthogonal Window Functions for Gabor Transforms

Xiang-Gen Xia

Abstract— **Gabor transforms have been recognized as useful tools in signal analysis. It is known that the solutions for the biorthogonal analysis window function $\gamma(t)$ given a synthesis window function $h(t)$ in Gabor transforms are not unique in general. Among these solutions, the minimum norm solution has already been given by Wexler and Raz in the discrete-time case and has been studied by Janssen, Ron and Shen, and Daubechies et al. in the continuous-time case. The minimum norm solution in the discrete-time case was also proved to be equal to the most orthogonal-like solution by Qian and Chen. In this note, we consider a general optimal-solution problem, where the minimum norm and the most orthogonal-like solutions are two special cases. We prove that these optimal solutions in many cases are equal. We also prove that it remains true in the continuous-time case.**

I. Introduction

Gabor transforms (or expansions) were first proposed by Gabor [1] to represent a signal in both the time and frequency domains. These have been attracting much attention recently, for example, in [2]–[16]. Let $h(t)$ be a synthesis window function so that

$$h_{m\alpha,m\beta}(t) = e^{-2\pi jm\alpha t}h(t-n\beta) \tag{1.1}$$

with α, β fixed and $m, n \in \mathbf{Z}$ forming a frame for $L^2(\mathbf{R})$ (see [5]–[11]). Let $s(t)$ be a signal; then

$$s(t) = \sum_{m,n} C_{m,n} h_{m\alpha,n\beta}(t) \tag{1.2}$$

where $C_{m,n}$ are constants. Notice that the coefficients $C_{m,n}$ in (1.2) are not unique in general, for example, in the case of $\alpha\beta < 1$. One way to find these $C_{m,n}$ from a signal $s(t)$ is to introduce an analysis biorthogonal window function $\gamma(t)$ so that

$$C_{m,n} = \int s(t)\gamma^*_{m\alpha,n\beta}(t)dt, \quad m, n \in \mathbf{Z} \tag{1.3}$$

where * means the complex conjugate. A relationship between the synthesis and the analysis window functions $h(t)$ and $\gamma(t)$ was established by Wexler and Raz [2] as

$$\int \gamma(t)h^*_{m/\alpha,n/\beta}(t)dt = \alpha\beta\delta_{m,0}\delta_{n,0}, \quad m, n \in \mathbf{Z}. \tag{1.4}$$

Known as the *Wexler-Raz identity* [5], it was rigorously proved by Janssen [7] and was generalized by Daubechies et al. [5]. Again, notice that the solution $\gamma(t)$ in (1.4) given an $h(t)$ is not unique.

The least squares choice for $C_{m,n}$ in (1.2) can be represented [5]–[11] by

$$C_{m,n} = \int s(t)\tilde{\gamma}^*_{m\alpha,n\beta}(t)dt, \quad m, n \in \mathbf{Z} \tag{1.5}$$

where $\tilde{\gamma} = S^{-1}_{h;\alpha,\beta}h$ and $S_{h;\alpha,\beta}$ is the frame operator

$$S_{h;\alpha,\beta}f = \sum_{m,n} h_{m\alpha,n\beta} \int f(t)h_{m\alpha,n\beta}(t)dt. \tag{1.6}$$

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The minimum norm solution of $\gamma(t)$ in (1.4) is denoted by $\hat{\gamma}(t)$. It was proved that $\tilde{\gamma} = \hat{\gamma}$ by Janssen [7], Ron and Shen [11], and Daubechies el al. [5] using the generalized Wexler–Raz identity. In other words, the least squares solution is equal to the minimum norm solution. A different formulation for $\hat{\gamma}$ was also found in [5]. We will see later, however, that the inversions of linear operators in $L^2(\mathbf{R})$ are needed. Wexler and Raz [2] and Qian and Chen [3], [4] recently studied discrete Gabor transforms, which can be described as follows.

Let $h[i]$ and $\gamma[i]$ be periodic discrete-time synthesis and analysis window functions with period L, respectively, so that

$$\sum_{i=0}^{L-1} h[i+mN]W_L^{-nMi}\gamma^*[i] = \delta_{m,0}\delta_{n,0}, \quad 0 \le m \le \Delta N - 1, \quad 0 \le n \le \Delta M - 1 \tag{1.7}$$

where $M, N, \Delta M, \Delta N$ are positive integers, and

$$W_L^{n\Delta Ni} = e^{2\pi n \Delta N i/L}.$$

Then

$$s[i] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C_{m,n}h_{m,n}[i] \tag{1.8}$$

$$C_{m,n} = \sum_{i=0}^{L-1} s[i]\gamma_{m,n}^*[i] \tag{1.9}$$

$$h_{m,n}[i] = h[i - m\Delta M]W_L^{n\Delta Ni}$$

$$\gamma_{m,n}[i] = \gamma[i - m\Delta M]W_L^{n\Delta Ni}$$

where ΔM and ΔN are, respectively, the time and frequency sampling interval lengths and M and N are the numbers of sampling points in the time and the frequency domains, respectively, $M \cdot \Delta M = N \cdot \Delta N = L$, $MN \ge L$ (or $\Delta M \cdot \Delta N \le L$). When $MN = \Delta M \cdot \Delta N = L$, it is the critical sampling case. Let

$$p \triangleq \Delta M \cdot \Delta N. \tag{1.10}$$

The relationship (1.7) between $h[i]$ and $\gamma[i]$ can be reexpressed by the following matrix form

$$H_{p\times L}\gamma_{L\times 1}^* = \mu_{p\times 1} \tag{1.11}$$

where $H_{p\times L}$ is a $p \times L$ matrix with its element at the $(m\Delta M + n)$th row and the ith column defined by

$$H(m\Delta M + n, i) = h[i+mN]W_L^{-nMi}, \quad 0 \le m < \Delta N, \quad 0 \le n < \Delta M, 0 \le i < L \tag{1.12}$$

$$\gamma_{L\times 1} = (\gamma[0], \gamma[1], \ldots, \gamma[L-1])^T \tag{1.13}$$

$$\mu_{p\times 1} = (1, 0, \ldots, 0)^T. \tag{1.14}$$

By a result in [9], the matrix $H_{p\times L}$ has full rank as long as the sequence h generates a discrete frame. In what follows, this is always assumed. Notice that, in the critical sampling case, i.e., $p = \Delta M \Delta N = L$, the solution of $\gamma_{L\times 1}$ in (1.11) is unique. In the oversampling case, i.e., $p < L$, $\gamma_{L\times 1}$ in (1.11) is not unique. The minimum norm solution $\hat{\gamma}[i]$ of (1.11), i.e.,

$$\min_{\gamma: H_{p\times L}\gamma_{L\times 1}^* = \mu_{p\times 1}} \sum_{i=0}^{L-1} |r[i]|^2$$

was given by Wexler–Raz [2] as

$$\hat{\gamma}_{L\times 1}^* = H_{p\times L}^\dagger (H_{p\times L}H_{p\times L}^\dagger)^{-1}\mu_{p\times 1} \tag{1.15}$$

where † means the complex conjugate transpose. Furthermore, Qian and Chen [3] proved that $\hat{\gamma}_{L\times 1}$ is also the most orthogonal-like solution of $\gamma_{L\times 1}$ in (1.11), as follows:

$$\sum_{i=0}^{L-1} |\hat{\gamma}[i] - h[i]|^2 = \min_{\gamma: H_{p\times L}\gamma_{L\times 1}^* = \mu_{p\times 1}} \sum_{i=0}^{L-1} |\gamma[i] - h[i]|^2. \tag{1.16}$$

In this note, we consider the following general optimization problem

$$\min_{\gamma: H_{p\times L}\gamma_{L\times 1}^* = \mu_{p\times 1}} \sum_{i=0}^{L-1} |\gamma[i] - (Ah)[i]|^2 \tag{1.17}$$

where A is a constant $L \times L$ matrix. It is clear that, when A is the 0 matrix, the solution of the optimization problem (1.17) is the minimum norm solution, and when A is the identity matrix I, the solution of (1.17) is the most orthogonal-like solution. Let γ^A be the solution of the optimization problem (1.17). We will prove that $\gamma^A = \hat{\gamma}$ when the constant matrix A commutes with the $L \times L$ matrix $H_{p\times L}^\dagger (H_{p\times L}H_{p\times L}^\dagger)^{-1}H_{p\times L}$. As a special case, $\gamma^A = \hat{\gamma}$ when $A = aI$ for any constant a. In this case, the optimization problem reduces to

$$\min_{\gamma: H_{p\times L}\gamma_{L\times 1}^* = \mu_{p\times 1}} \sum_{i=0}^{L-1} |\gamma[i] - ah[i]|^2.$$

In other words, all the above optimal solutions for the analysis biorthogonal window functions for different constants a are equal to the minimum norm solution.

When some of the eigenvalues of $H_{p\times L}H_{p\times L}^\dagger$ are close to zero, the optimal solution $\hat{\gamma}$ in (1.15) is not stable. Qian and Chen [4] used the singular value decomposition approach to handle this ill posedness. In this note, we use the regularization approach. With this approach, we are able to estimate the error of the regularized solution and the true solution.

For continuous-time Gabor transforms, we will prove that the minimum norm solution $\hat{\gamma}(t)$ in (1.4) is also equal to the most orthogonal-like solution in (1.4). Specifically,

$$\int |\hat{\gamma}(t) - Ah(t)|^2 dt = \min_{\gamma \text{ in } (1.4)} \int |\gamma(t) - (Ah)(t)|^2 dt \tag{1.18}$$

where A is an operator on $L^2(\mathbf{R})$. See Janssen [8] for a study of A as the identity operator.

This correspondence is organized as follows. In Section II, we discuss discrete Gabor transforms. In Section III, we study continuous Gabor transforms.

II. Optimal Biorthogonal Window Functions for Discrete Gabor Transforms

In this section, we consider the optimization problem (1.17) where $h[i]$ is given. There are an orthogonal matrix $Q_{L\times L}$ and an invertible lower triangular matrix $R_{p\times p}$ such that

$$H_{p\times L} = (R_{p\times p}, 0_{p\times(L-p)})Q_{L\times L} \tag{2.1}$$

where $0_{p\times(L-p)}$ is the $p \times (L-p)$ all-zero matrix.

Let $x^A[i] = \gamma[i] - (Ah)[i]$. Then the optimization problem (1.17) is equivalent to

$$\min_{x^A: H_{p\times L}(x_{L\times 1}^A)^* = \eta_{p\times 1}} \sum_{i=0}^{L-1} |x^A[i]|^2 \tag{2.2}$$

where $\eta_{p\times 1} = \mu_{p\times 1} - H_{p\times L}A^*h_{L\times 1}^*$ and $h_{L\times 1} = (h[0], \ldots, h[L-1])^T$. The solution of (2.2) is the minimum norm solution of the system $H_{p\times L}(x_{L\times 1}^A)^* = \eta_{p\times 1}$, which is

$$(\hat{x}_{L\times 1}^A)^* = H_{p\times L}^\dagger (H_{p\times L}H_{p\times L}^\dagger)^{-1}\eta_{p\times 1}.$$

Therefore, the solution γ^A for the optimization problem (1.17) is

$$\begin{aligned}(\gamma_{L\times 1}^A)^* &= (\hat{x}_{L\times 1}^A)^* + A^*h_{L\times 1}^* \\ &= A^*h_{L\times 1}^* - H_{p\times L}^\dagger (H_{p\times L}H_{p\times L}^\dagger)^{-1}H_{p\times L}A^*h_{L\times 1}^* \\ &\quad + H_{p\times L}^\dagger (H_{p\times L}H_{p\times L}^\dagger)^{-1}\mu_{p\times 1}.\end{aligned} \tag{2.3}$$

When $H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}A^* = A^*H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}$, we check the expression

$$H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}h^*_{L\times 1}.$$

By the form of $H_{p\times L}$ in (1.12), $h^*_{L\times 1}$ is the first column of $H^\dagger_{p\times L}$. Since $R_{p\times p}$ is lower triangular,

$$h^*_{L\times 1} = Q^\dagger_{L\times L}(r_{1,1}, 0, \ldots, 0)^\dagger \tag{2.4}$$

where $r_{1,1}$ is the element at the first row and the first column of the matrix $R_{p\times p}$. By using (2.1) and (2.4), it is not hard to see

$$\begin{aligned}&H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}h^*_{L\times 1}\\ &\quad= Q^\dagger_{L\times L}(r_{1,1}, 0, \ldots, 0)^\dagger = h^*_{L\times 1}.\end{aligned}$$

Going back to (2.3), we have proved the following theorem.

Theorem 1: The optimal biorthogonal window function γ^A solving the optimization problem (1.17) can be expressed by

$$(\gamma^A_{L\times 1})^* = H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}\mu_{p\times 1} \tag{2.5}$$

when matrix A commutes with matrix $H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}$, as in

$$\begin{aligned}&H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}A^*\\ &\quad= A^*H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}.\end{aligned}$$

As a consequence of Theorem 1, the minimum norm solution (when $A = 0$) and the most orthogonal-like solution (when $A = I$) of (1.11) or (1.7) are equal. There are many cases of such a matrix A in Theorem 1 that are not trivial, i.e., $A = aI$ for constants a. Let

$$B \triangleq H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L})^{-1}H_{p\times L}.$$

Then, B is Hermitian and can be diagonized by a unitary matrix U: $B = U^\dagger \Lambda U$ where Λ is a diagonal matrix. Thus, all matrixes$A = U^\dagger\Lambda_1 U$ with diagonal matrixes Λ_1 satisfy the condition in Theorem 1, i.e., they commute with matrix B.

Consider the case when some of the eigenvalues of the matrix $H_{p\times L}H^\dagger_{p\times L}$ are close to zero. In this case, the inverse of $H_{p\times L}H^\dagger_{p\times L}$ is unstable. This may occur when the synthesis window function $h[i]$ has Gaussian shape. Qian and Chen [4] studied this problem by using the singular value decomposition method. We now want to use the regularization method. This method enables us to estimate the error between the regularized solution and the true solution, which is continuously dependent on the regularization parameter ϵ.

From Theorem 1, the optimal solution γ^A is of the form (2.5). We propose the following regularized solution:

$$(\gamma^A_{L\times 1}(\epsilon))^* = H^\dagger_{p\times L}(H_{p\times L}H^\dagger_{p\times L} + \epsilon I_{p\times p})^{-1}\mu_{p\times 1} \tag{2.6}$$

where $\epsilon > 0$ is an arbitrary constant and $I_{p\times p}$ is the $p\times p$ identity matrix. From [15], we have the following error estimate:

$$\|\gamma^A_{L\times 1}(\epsilon) - \gamma^A_{L\times 1}\| \le C\epsilon^{1/2} \tag{2.7}$$

where C is a positive constant.

III. Optimal Biorthogonal Window Functions for Continuous Gabor Transforms

In this section, we study optimal analysis biorthogonal window functions $\gamma(t)$ given a synthesis window function $h(t)$ as in (1.4). Before going through the details, we first introduce some notations. For two signals $f, g \in L^2(\mathbf{R})$, $\langle f, g\rangle$ denotes the inner product of f and g in $L^2(\mathbf{R})$ in the usual sense. $T_{g;\alpha,\beta}$ denotes the following linear operator from $L^2(\mathbf{R})$ to $l^2(\mathbf{Z}^2)$:

$$T_{g;\alpha,\beta}f = (\langle f, g_{m\alpha,n\beta}\rangle)_{m,n\in\mathbf{Z}}.$$

$T^*_{g;\alpha,\beta}$ denotes the dual of $T_{g;\alpha,\beta}$, which maps signals from $l^2(\mathbf{Z}^2)$ to $L^2(\mathbf{R})$as follows:

$$T^*_{g;\alpha,\beta}C = \sum_{m,n} C_{m,n}g_{m\alpha,n\beta}, \text{ for } C = (C_{m,n})_{m,n\in\mathbf{Z}}.$$

Let $e_{0,0} = (C_{m,n})_{m,n\in\mathbf{Z}}$ denote the sequence in $l^2(\mathbf{Z}^2)$ where $C_{0,0} = 1$ and $C_{m,n} = 0$ for other m, n.

With the above notations, the Wexler–Raz identity (1.4) can be rewritten as

$$T_{h;1/\beta,1/\alpha}\gamma = \alpha\beta e_{0,0}. \tag{3.1}$$

The minimum norm solution problem is

$$\min_{\gamma: T_{h;1/\beta,1/\alpha}\gamma = \alpha\beta e_{0,0}} \int |\gamma(t)|^2 dt. \tag{3.2}$$

The solution $\hat{\gamma}$ for (3.2) was found in [5] as

$$\hat{\gamma} = \alpha\beta T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}e_{0,0}. \tag{3.3}$$

Similar to discrete Gabor transforms discussed in Section II, we consider the following more general optimization problem instead of (3.2):

$$\min_{\gamma: T_{h;1/\beta,1/\alpha}\gamma = \alpha\beta e_{0,0}} \int |\gamma(t) - Ah(t)|^2 dt \tag{3.4}$$

where A is an operator on $L^2(\mathbf{R})$. Similar to the discussion in Section II, we reduce (3.4) into (3.2) by letting $x^A(t) = \gamma(t) - Ah(t)$, as follows:

$$\min_{x^A: T_{h;1/\beta,1/\alpha}x^A = \eta} \int |x^A(t)|^2 dt \tag{3.5}$$

where $\eta = \alpha\beta e_{0,0} - AT_{h;1/\beta,1/\alpha}h$. The solution $\hat{x}^A$ for (3.5) is

$$\hat{x}^A = \alpha\beta T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}\eta.$$

Therefore, the solution γ^A for optimization problem (3.4) can be formulated as

$$\begin{aligned}\gamma^A &= \hat{x}^A + Ah\\ &= Ah - T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}T_{h;1/\beta,1/\alpha}Ah\\ &\quad + \alpha\beta T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}e_{0,0}.\end{aligned}$$

It is clear that

$$h = h_{0/\beta,0/\alpha} = T^*_{h;1/\beta,1/\alpha}e_{0,0}. \tag{3.6}$$

When operator A commutes with operator $T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}T_{h;1/\beta,1/\alpha}$, by using (3.6) we can easily see that

$$Ah - AT^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}T_{h;1/\beta,1/\alpha}h = 0.$$

This proves the following theorem.

Theorem 2: The solution γ^A for optimization problem (3.4) can be represented as

$$\gamma^A = \alpha\beta T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}e_{0,0} \tag{3.7}$$

when operator A commutes with operator $T^*_{h;1/\beta,1/\alpha}(T_{h;1/\beta,1/\alpha}T^*_{h;1/\beta,1/\alpha})^{-1}T_{h;1/\beta,1/\alpha}$.

Theorem 2 implies that the minimum norm solution $\hat{\gamma} = \gamma^0$ ($A = 0$) in (3.3) is also equal to the most orthogonal-like solution γ^I ($A = I$) where I denotes the identity operator. Therefore, the least squares solution, the minimum norm solution, and the most orthogonal-like solution are equal.

IV. CONCLUSION

In this correspondence, we studied general optimal solutions for the biorthogonal analysis window functions given a synthesis window function in both discrete and continuous Gabor transforms. The common minimum norm solutions and the most orthogonal-like solutions are just two special cases. We proved that the optimal solutions in many cases are equal.

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Analytical Formulae for Reconstruction of Certain Discrete Signals from Phase Level and Line Crossings

Andrew E. Yagle

***Abstract*— We provide simple and explicit formulae for reconstructing any member of a class of discrete-time signals from the frequencies at which its Fourier phase crosses any specific level of constant phase or a linear-phase line with integer slope, provided that the number of crossings equals the length of the signal support. Unlike previous closed-form solutions, solution of an ill-conditioned system of linear equations is not required. The associated uniqueness results reduce, in special cases, to previous results for reconstruction from Fourier transform real and imaginary part zero crossings.**

I. INTRODUCTION

The problem of reconstructing a signal from its Fourier phase has been studied extensively over the last fifteen years. Reference [1] gives the basic uniqueness results for reconstructing a 1-D discrete-time finite-support signal from its phase, and two algorithms for this reconstruction were proposed. One algorithm was called a *closed-form solution*, although it required the solution of a large and ill-conditioned linear system of equations (see (2) below). The other was an alternating-projections (AP) algorithm, in which the support and given phase value constraints were alternately imposed in the time and frequency domains. This algorithm was deemed to be preferable to the first.

In [2], another uniqueness result requiring a certain matrix to have full rank was obtained. In [3], a time-domain approach to reconstruction from specific phase values resulted in a Toeplitz-plus-Hankel linear system of equations. In [4], the closely related problem of reconstruction from one bit of Fourier phase (sign of the real part of the Fourier transform) was briefly discussed, and a solution procedure was verbally outlined. Segmentation of the Fourier phase was proposed in [3] and [5]. Extension to the 2-D case was made in [6] and [7], and the effects of noise were studied in [8]. A review of the dual problem of reconstruction of 2-D bandlimited signals from zero crossings can be found in [9].

The main application of reconstruction from phase is in blind deconvolution of an unknown symmetric or Hermitian blurring function to determine an unknown signal, when only the supports of both unknown functions are known [10],[11]. We briefly review this below. Other applications are listed in [3],[4], and [7].

Two main approaches have been proposed to reconstruct signals from their phase. One approach [1]–[3] requires the solution of a large and generally ill-conditioned system of linear equations. Serious problems can arise, however, due to long computation times, storage, access time, and roundoff error from poor conditioning of the problem. The other approach [1]–[11] is to use an AP algorithm similar to the one proposed in [1]. Such algorithms are generally faster than solving the linear system of equations, but thousands of iterations may be necessary to obtain a good approximation to the solution.

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