Delay Doppler Transform

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***Abstract*—This letter is to introduce delay Doppler transform (DDT) for a time domain signal. It is motivated by the recent studies in wireless communications over delay Doppler channels that have both time and Doppler spreads, such as, satellite communication channels. We present some simple properties of DDT as well. The DDT study may provide insights of delay Doppler channels.**

***Index Terms*—OFDM, VOFDM, OTFS, delay Doppler transform (DDT).**

I. Introduction

THE SUCCESS of Starlink has re-generated world wide interest on satellite communications. A special characteristic of satellite communications is that its channel is not only time spread but also Doppler spread for wideband transmissions [1], [2]. To deal with such channels, there have been many studies, for example [1], [3], [4] for channel estimations. A recent popular topic is orthogonal time frequency space (OTFS) modulation [5] that has been shown identical to vector OFDM (VOFDM) [6], [7] in [8], [9], [10], [12], at least, from the transmission side.

For a delay Doppler channel, see for example [1], [3], [4], at time delay τ, let

$$h(\tau, t) = g(\tau) e^{-j\Omega(\tau)t} \tag{1}$$

be its channel response with Doppler shift $\Omega(\tau)$ that is a function of time delay τ. This channel means that the path $h(\tau, t)$ of time delay τ has Doppler shift $\Omega(\tau)$ and in general, different paths at different time delays may have different Doppler shifts.

Let $s(t)$ be a transmitted signal. Then, the received signal $y(t)$ at time t is

$$\begin{aligned} y(t) &= \int h(\tau, t) s(t-\tau) d\tau + w(t) \\ &= \int g(\tau) s(t-\tau) e^{-j\Omega(\tau)t} d\tau + w(t), \end{aligned} \tag{2}$$

where $w(t)$ is the additive noise.

When the Doppler shift function $\Omega(\tau)$ in (2) is a constant Ω that does not depend on τ, i.e., the trivial Doppler spread case, it means that all the channel responses at all the time delays have the same Doppler shift Ω. In this case, this Doppler shift can be compensated at either transmitter or receiver and the compensated channel then becomes a time spread only channel. Otherwise, different multipaths have different Doppler shifts and it is called non-trivial Doppler spread case. An example of such a non-trivial Doppler spread case is when the reflection multipaths from different moving objects with different locations move with different velocities.

Manuscript received 12 March 2024; accepted 1 April 2024. This work was supported in part by the National Science Foundation (NSF) of USA under Grant CCF 2246917. The associate editor coordinating the review of this article and approving it for publication was J. Wang.

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Digital Object Identifier 10.1109/LWC.2024.3384529

II. Definition

Motivated from the above delay Doppler channel model, we define delay Doppler transform (DDT) below.

Definition 1: Let $g(t)$ be a window function and $s(t)$ be a signal. The DDT of $s(t)$ is defined as

$$DDT_s(t, \Omega) = \int s(\tau) g(t-\tau) e^{-j\Omega(\tau)t} d\tau, \tag{3}$$

where $\Omega(\tau)$ is a function of τ.

When function $\Omega(\tau)$ in (3) is linear in terms of τ, i.e, $\Omega(\tau) = \Omega\tau$ for a constant Ω, the above definition becomes

$$DDT_s(t, \Omega) = \int s(\tau) g(t-\tau) e^{-j\Omega\tau t} d\tau. \tag{4}$$

In this case, we call Ω as the Doppler shift rate (or frequency rate) of the transform (or the channel). The DDT in (4) measures signal $s(t)$ by window function $g(-t)$ across its all time shifts and Doppler shifts with a non-zero Doppler shift rate Ω. It is different from the short time Fourier transform (STFT) of $s(t)$ with window function $g(t)$, which is

$$STFT_s(t, \Omega) = \int s(\tau) g(\tau - t) e^{-j\Omega\tau} d\tau. \tag{5}$$

STFT is to measure $s(t)$ by a given window function $g(t)$ across its all time and frequency shifts. The above DDT is much different from Zak transform [18] that does not tell when signal frequency changes as a typical joint time-frequency analysis technique does [17].

Note that in the scenario when farther reflectors move faster, their corresponding reflection mutlipaths may have their Doppler shifts approximately linear in terms of time t as above. Another note is that in the above DDT, window function $g(-t)$ is used, which is for notational convenience to better match the above delay Doppler channel.

When function $\Omega(\tau)$ in (3) is constant, the above definition becomes

$$DDT_s(t, \Omega) = e^{-j\Omega t} \int s(\tau) g(t-\tau) d\tau, \tag{6}$$

where Ω is constant and does not depend on the time delay τ, i.e., a trivial Doppler spread. In this case, it is clear that after the compensation of the common Doppler shift at transmitter, the DDT becomes the convolution, i.e., the channel is time spread only.

It is known that OFDM (or Fourier transform) converts a time spread only channel to multiple non-time spread subchannels. To improve the transmission signal spectrum, a pulse (or window) with better spectrum than the rectangular pulse is added, which is the generalized frequency division multiplexing (GFDM) [13]. Another purpose to use a window in GFDM is to limit the OFDM block size, i.e., to have a smaller block size than the conventional OFDM to adapt to a time varying channel. GFDM is different from VOFDM (or OTFS), unless the vector size in VOFDM is 1 and then in this case, VOFDM returns to OFDM. VOFDM converts a time spread only channel to multiple vector subchannels where there is no time spread (or intersymbol interference (ISI)) across vector subchannels, while there is ISI inside each vector subchannel. OFDM corresponds to discrete Fourier transform (DFT) filterbank [16], VOFDM corresponds to vector DFT filterbank [7], and GFDM corresponds to discrete Gabor transform (DGT) [14], [15] (or STFT with a given window function).

From (4), the DDT, that corresponds to a non-trivial but simply a linear Doppler spread in terms of time delay, is different from all the existing joint time-frequency transforms/distributions in the literature. This means that the existing joint time-frequency transforms may not be helpful to deal with non-trivial Doppler spread and time-spread channels. It also implies that it is not possible to well compensate non-trivial Doppler spread at either transmitter or receiver. So, neither GFDM nor VOFDM (OTFS) can compensate a non-trivial Doppler spread well. However, since for VOFDM (OTFS) it is demodulated vector-wisely and in the mean time due to its inherited structure, VOFDM is able to collect multipath diversity for time varying channels in general [11]. Thus, VOFDM (OTFS) performs better than OFDM over delay Doppler channels. For more details, see [10], [11].

From (1), one can see that the delay Doppler channel response function is a special case of general two dimensional delay Doppler channel response function $h(\tau, t)$ that can be an arbitrary two dimensional function of time delay variable τ and time variable t. Thus, it is even more difficult to compensate the Doppler spread in a more general two dimensional delay Doppler channel.

For the inverse DDT, when the window function $g(t)$ is known, $s(t)$ can be obtained from the deconvolution of $DDT_s(t, \Omega)$ at $\Omega = 0$.

III. Properties

We now present some simple properties of the DDT in (4). We first consider the DDT of a time shifted signal $s(t - t_0)$ with time shift t_0. Then, from (4) we have

$$
\begin{aligned}
&DDT_{s(t-t_0)}(t, \Omega) \\
&\quad = \int s(\tau - t_0) g(t - \tau) e^{-j\Omega\tau t} d\tau \\
&\quad = \int s(\tau - t_0) g(t - t_0 - (\tau - t_0)) \\
&\quad \cdot e^{-j\Omega(\tau-t_0)(t-t_0)} e^{-j\Omega(\tau-t_0)t_0 - j\Omega t_0 t} d\tau \\
&\quad = \int s(\tau - t_0) e^{-j\Omega(\tau-t_0)t_0} g(t - t_0 - (\tau - t_0)) \\
&\quad \cdot e^{-j\Omega(\tau-t_0)(t-t_0)} d\tau e^{-j\Omega t_0 t} \\
&\quad = DDT_{s(t)e^{-j\Omega t_0 t}}(t - t_0, \Omega) e^{-j\Omega t_0 t}.
\end{aligned} \tag{7}
$$

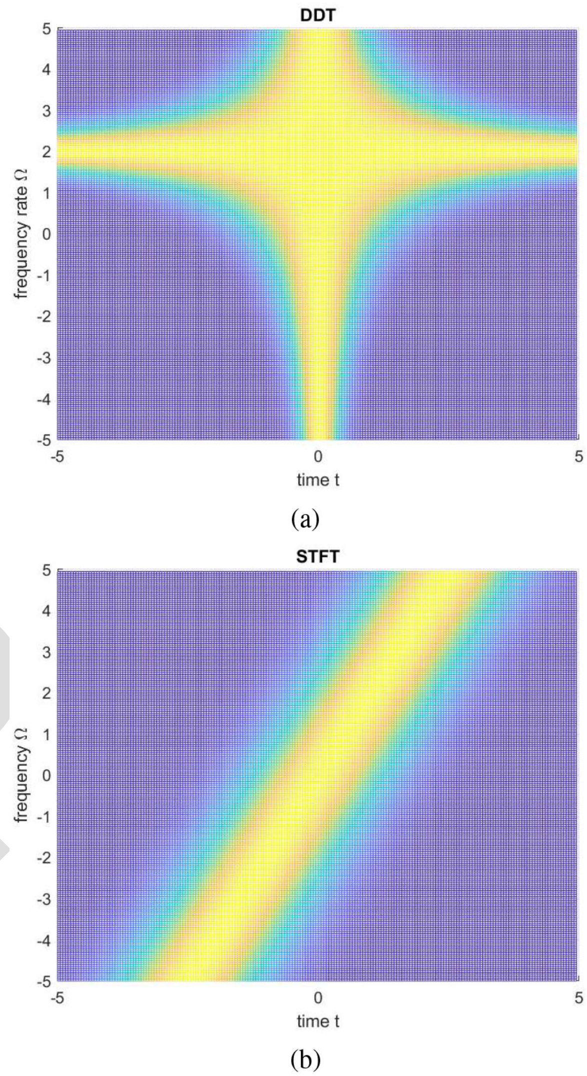


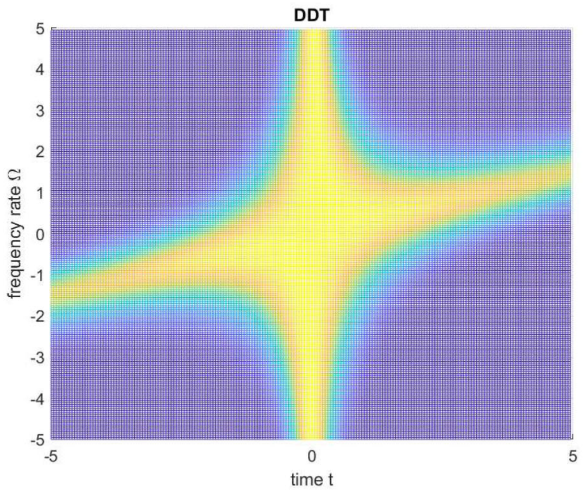
Fig. 1. The DDT and STFT of signal $s_1(t) = e^{jt^2}$: (a) DDT; (b) STFT. AQ2

We know that the Fourier transform or STFT of a time shifted signal is that of the orignal signal modulated in frequency. However, from (7), one can see that it is different from the Fourier transform or STFT in the sense that the DDT of a time shifted $s(t)$ is the time shifted and additionally modulated DDT of the modulated $s(t)$.

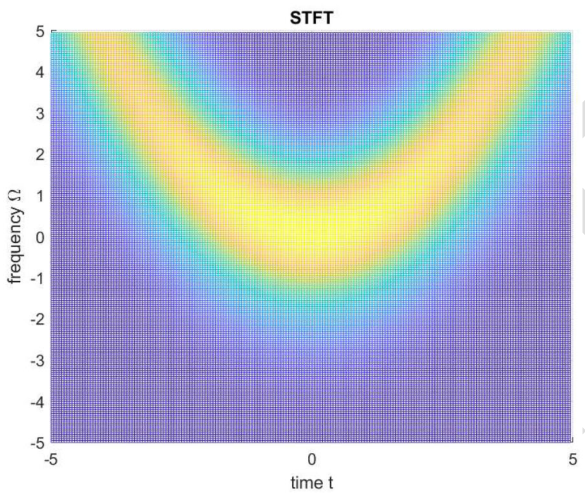
For the delay and Doppler channel (2), the received signal can be represented by the DDT of the transmitted signal $s(t)$ with the channel response amplitude function $g(t)$ as the window function below:

$$y(t) = DDT_s(t, -\Omega) e^{-j\Omega t^2} + w(t). \tag{8}$$

From (8), one can see that the signal part of the received signal is the linear chirp modulated DDT of the transmitted signal

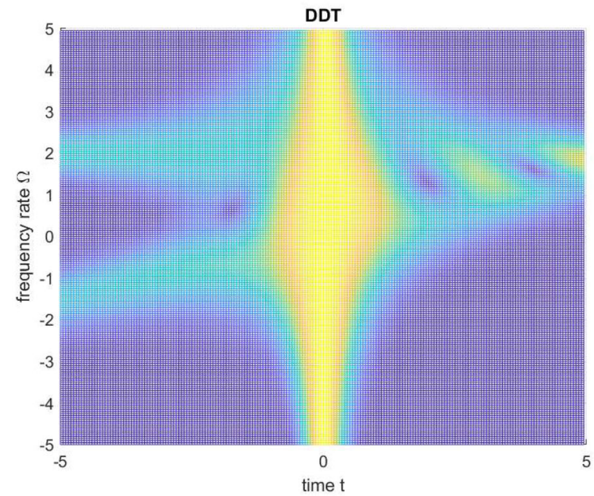


(a)

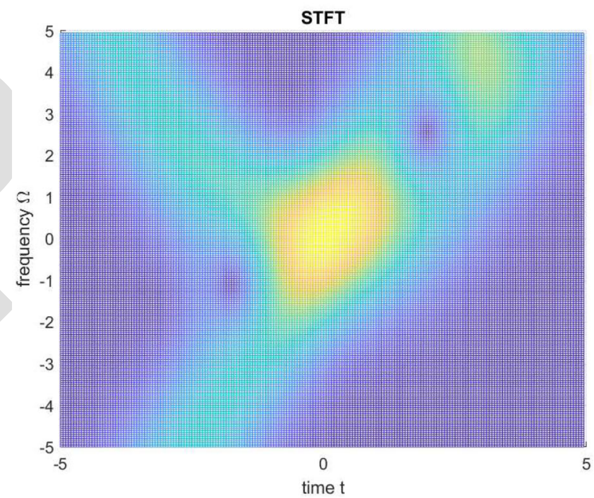


(b)

Fig. 2. The DDT and STFT of signal $s_2(t) = e^{jt^3/10}$: (a) DDT; (b) STFT.



(a)



(b)

Fig. 3. The DDT and STFT of signal $s_3(t) = s_1(t) + s_2(t) = e^{jt^2} + e^{jt^3/10}$: (a) DDT; (b) STFT.

evaluated at the negative Doppler shift rate, i.e., $-\Omega$. In other words, the dechirped received signal $y(t)e^{j\Omega t^2}$ is a DDT of the transmitted signal. This implies that the study of DDT is important for the communication over the delay Doppler channel (2).

We next consider a transmitted signal in a communication system:

$$s(t) = \sum_n s_n p(t - nT), \tag{9}$$

where s_n are the information symbols to transmit, $p(t)$ is the pulse, and T is the symbol duration. Then, using the property (7), its DDT is

$$DDT_s(t, \Omega) = \sum_n s_n DDT_{p_n}(t - nT, \Omega) e^{-jnTt\Omega}, \tag{10}$$

where p_n is the modulated $p(t)$:

$$p_n = p_n(t) = p(t)e^{-jnT\Omega t}. \tag{11}$$

From (8) and (10), at the receiver we have the following new channel:

$$y'(t) = \sum_n s_n DDT_{p_n}(t - nT, -\Omega) e^{jnTt\Omega} + w'(t), \tag{12}$$

where $y'(t) = y(t)e^{j\Omega t^2}$ and $w'(t) = w(t)e^{j\Omega t^2}$. Since the above dechirping is a unitary operation, it does not change the received signal or the noise property. The above DDT based receive signal model (12) might provide insights in designing pulses $p(t)$ in better dealing with delay Doppler channels in communications systems. It might have applications in radar waveform designs to deal with multiple maneuvering moving objects.

IV. SIMULATIONS

We now see some plots of the DDT in (4) and STFT in (5) for some simple signals. The window function we use is a Gassian function $g(t) = e^{-t^2}$. Three signals are tested. The first is a linear chirp $s_1(t) = e^{jt^2}$, the second is a quadratic chirp $s_2(t) = e^{jt^3/10}$, and the third is their sum, i.e., $s_3(t) = s_1(t) + s_2(t)$. All of them are supported on $[-10,10]$. The magnitudes of DDT and STFT of these three signals in the region $(t, \Omega) \in [-5, 5] \times [-5, 5]$ are shown in Figs. 1-3.

We know that the STFT roughly tells the joint time frequency distribution property for a signal, although its resolution may not be as high as those of non-linear time frequency distributions [17], such as Wigner-Ville distribution. From these figures, we find that the DDT of a signal is much different from a joint time frequency distribution, which may help to understand a delay Doppler channel more.

V. Conclusion

In this letter, we introduced delay Doppler transform (DDT) for a signal. It was motivated from the recent interest in wireless communications over delay Doppler channels, such as satellite channels. We also provided some simple properties about DDT. One can see that DDT is different from all the existing joint time frequency analysis techniques and may provide more insights for delay Doppler channels, such as more characteristics for a radio map. From the study in this letter, we may see that for a non-trivial Doppler spread channel, no existing modulation scheme (neither VOFDM/OTFS nor GFDM) can deal with it well.

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