1. (30 pts) Consider the prediction of a real, stationary process $\{x(n)\}$. We wish to investigate the forward and backward two–step prediction of this process. The observation vectors are thus $\mathbf{x}_f(n) = [x(n-1), x(n-2)]^T$ and $\mathbf{x}_b(n) = [x(n), x(n-1)]^T$, respectively, while the desired signals in each case are $x(n)$ and $x(n-3)$.

- (10 pts) Derive the weight vector $\mathbf{w}_f$ that minimizes the forward prediction MSE.
- (10 pts) Derive the weight vector $\mathbf{w}_b$ that minimizes the backward prediction MSE.
- (5 pts) Derive a relationship between $\mathbf{w}_f$ and $\mathbf{w}_b$.
- (5 pts) Derive the minimum MSE for each case and indicate how the minimum MSE values are related.

2. (25 pts) Consider the AR process

$$x(n) = ax(n-2) + v(n)$$

where $v(n)$ is white with variance σ^2.

- (5 pts) What bounds on a will guarantee $x(n)$ to be stable?
- (10 pts) Derive a general expression for $r(k) = E\{x(n)x(n-k)\}$.
- (5 pts) Derive a general expression for the weight of a single tap predictor with lag k,

$$\hat{x}(n) = w_k x(n-k).$$

- (5 pts) Derive a general expression for the error of this predictor.

3. (25 pts) Consider the linear estimate $\hat{d}(n) = \mathbf{w}^T\mathbf{x}(n)$ of a desired signal $d(n)$ that is statistically related to the observation vector $\mathbf{x}(n)$. Suppose the performance criteria to be minimized is $J_K(\mathbf{w}) = E\left[(d(n) - \hat{d}(n))^{2K}\right]$, where $K \in \{1, 2, \ldots\}$. Derive a LMS type update that optimizes $\mathbf{w}$ based on this performance criteria.

4. (20 pts) Prove the following:

- (10 pts) The determinant relation

$$\det(\mathbf{R}) = \prod_{i=0}^{N} \lambda_i$$

  holds for the correlation matrix $\mathbf{R}$ of a stationary random process, where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of $\mathbf{R}$.

- (10 pts) The mean square error of a transversal filter can be written as

$$J(\mathbf{w}) = J_{\min} + (\mathbf{w} - \mathbf{w}_o)^H \mathbf{R}(\mathbf{w} - \mathbf{w}_o),$$

  where $J_{\min}$ is the minimum MSE achieved when the Weiner solution weights, $\mathbf{w}_o$, are utilized.