University of Delaware

ELEG–636 Midterm, April 7, 2009

Closed Book Test
Use Of Books, Written Materials, Calculators,
And All Electronic Devices Prohibited.

1. [30 pts] Probability:

   (a) [15 pts] Prove the Bienayme inequality, which is a generalization of the Tchebycheff inequality,

   $$Pr\{|X-a| \geq \epsilon\} \leq \frac{E\{|X-a|^n\}}{\epsilon^n}$$

   for arbitrary a and distribution of X.

   (b) [15 pts] Consider the uniform distribution over $[-1,1]$.

      i. [10 pts] Determine the moment generating function for this distribution.

      ii. [5 pts] Use the moment generating function to generate a simple expression for the k'^{th} moment, m_k.

Answer:

(a)

$$\begin{aligned} E\{|x-a|^n\} &= \int_{-\infty}^{\infty} |x-a|^n f_x(x)dx \geq \int_{x-a|\geq\epsilon} |x-a|^n f_x(x)dx \geq \int_{x-a|\geq\epsilon} \epsilon^n f_x(x)dx \\ &= \epsilon^n Pr\{|x-a| \geq \epsilon\} \Rightarrow Pr\{|X-a| \geq \epsilon\} \leq \frac{E\{|X-a|^n\}}{\epsilon^n} \end{aligned}$$

(b)

$$\begin{aligned} \Phi(s) =& \frac{1}{2}\int_{-1}^{1} e^{sx}dx = \begin{cases} \frac{1}{2s}(e^s - e^{-s}) & s \neq 0 \\ 1 & s = 0 \end{cases} \\ \Rightarrow E\{x^k\} =& \left.\frac{d^k\Phi(s)}{d^k s}\right|_{s=0} \\ E\{x\} =& \left.\frac{d\Phi(s)}{ds}\right|_{s=0} = \left.\frac{1}{2s}(e^s + e^{-s}) - \frac{1}{2s^2}(e^s - e^{-s})\right|_{s=0} \\ =& \left.\frac{1}{2}(e^s - e^{-s}) - \frac{1}{4}(e^s - e^{-s})\right|_{s=0} = 0 \end{aligned}$$

Repeat the differentiation, limit (l'Hpital's rule) process. The analytical solution is simpler:

$$E\{x^k\} = \frac{1}{2}\int_{-1}^{1} x^k dx = \frac{1-(-1)^{k+1}}{2(k+1)} = \begin{cases} 0 & k = 1,3,5,\ldots \\ \frac{1}{k+1} & k = 0,2,4,\ldots \end{cases}$$

2. [35 pts] Let $v[n]$ be white noise with variance $\sigma_v^2 = 1$ and set

$$x[n] = v[n] + \frac{1}{2}v[n-1].$$

(a) [10 pts] Determine the mean and correlation expressions for the output.

(b) [25 pts] Suppose $x[n]$ is passed through a transmission channel with the received signal modeled as

$$y[n] = x[n] + \eta[n]$$

where $\eta[n]$ is white noise, independent of $v[n]$, with variance $\sigma_\eta^2 = 2$. Define the two element observation vector $\mathbf{y}[n] = [y[n], y[n-1]]^T$.

i. [15 pts] Determine the matched filter that best recovers $x[n]$, i.e., the weight vector $\mathbf{w} = [w_1, w_2]^T$ that maximizes the SNR of the output $\hat{x}[n] = \mathbf{w}^T\mathbf{y}[n]$.

ii. [10 pts] Determine the resulting SNR when the optimal $\mathbf{w}$ is employed.

Answer:

(a) $E\{x[n]\} = E\{v[n]\} + \frac{1}{2}E\{v[n-1]\} = 0$ and

$$\begin{aligned} r_x[k] =& E\{x[n]x[n-k]\} = E\left\{\left(v[n] + \frac{1}{2}v[n-1]\right)\left(v[n-k] + \frac{1}{2}v[n-k-1]\right)\right\} \\ =& \delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] + \frac{1}{4}\delta[k] = \frac{5}{4}\delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \end{aligned}$$

(b) For $N = 2$

$$\mathbf{R}_x = \begin{bmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\left|\begin{bmatrix} 5-\lambda' & 2 \\ 2 & 5-\lambda' \end{bmatrix}\right| = (5-\lambda')^2 - 4 = \lambda'^2 - 10\lambda' + 21 = (\lambda'-3)(\lambda'-7) = 0 \Rightarrow \lambda' = 3, 7$$

Thus $\lambda = \frac{\lambda'}{4} = \frac{3}{4}, \frac{7}{4}$ and $\mathbf{q}_{\max}$ is found from $(\mathbf{R}_x - \lambda_{\max}\mathbf{I})\mathbf{q}_{\max} = \mathbf{0}$, or

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\mathbf{q}_{\max} = \mathbf{0} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{q}_{\max} = \mathbf{0} \Rightarrow \mathbf{q}_{\max} = \frac{1}{\sqrt{2}}[1,1]^T$$

and $\mathbf{w} = \mathbf{q}_{\max} = \frac{1}{\sqrt{2}}[1,1]^T$. The resulting SNR is $\frac{\lambda_{\max}}{\sigma_\eta^2} = \frac{7}{8}$.

3. [35 pts] Let $Z = X+N$, where X and N are independent with distributions $N \sim \mathcal{N}(0, \sigma_N^2)$ and $f_X(x) = \frac{1}{2}\delta(x-2) + \frac{1}{2}\delta(x+2)$.

   (a) [15 pts] Determine the MAP, MS, MAE, and ML estimates for X in terms of Z.

   (b) [10 pts] Determine the bias of each estimate, i.e., determine whether or not each estimate is biased.

   (c) [10 pts] Determine the variances of the estimates.

   Answer:

   (a) Since X and N are independent, $f_Z(z) = f_X(z) * f_N(z) = \frac{1}{2}\mathcal{N}(-2, \sigma_N^2) + \frac{1}{2}\mathcal{N}(2, \sigma_N^2)$. Also

$$
\begin{aligned}
f_{Z|X}(z|x) =& \mathcal{N}(x, \sigma_N^2) \\
\hat{x}_{ML} =& \arg\max_x f_{Z|X}(z|x) = z \\
f_{X|Z}(x|z) =& \frac{f_{Z|X}(z|x) f_X(x)}{f_Z(z)} = \frac{\mathcal{N}(x, \sigma_N^2)(\delta(x-2) + \delta(x+2))}{2 f_Z(z)} \\
\hat{x}_{MAP} =& \arg\max_x f_{X|Z}(x|z) = \begin{cases} 2 & z > 0 \\ -2 & z < 0 \end{cases} \\
\hat{x}_{MS} =& \int_{-\infty}^{\infty} x f_{X|Z}(x|z) dx = \frac{1}{f_Z(z)} \int_{-\infty}^{\infty} x f_{Z|X}(z|x) f_X(x) dx \\
=& \frac{\left(2\mathcal{N}(2, \sigma_N^2)|_{x=z} - 2\mathcal{N}(-2, \sigma_N^2)|_{x=z}\right)}{2 f_Z(z)} \\
=& 2\frac{\mathcal{N}(2, \sigma_N^2)|_{x=z} - \mathcal{N}(-2, \sigma_N^2)|_{x=z}}{\mathcal{N}(2, \sigma_N^2)|_{x=z} + \mathcal{N}(-2, \sigma_N^2)|_{x=z}} \\
\frac{1}{2} =& \int_{-\infty}^{\hat{x}_{MAE}} f_{X|Z}(x|z) dx = \frac{1}{f_Z(z)} \int_{-\infty}^{\hat{x}_{MAE}} f_{Z|X}(z|x) f_X(x) dx \\
\Rightarrow \int_{-\infty}^{\hat{x}_{MAE}} f_{Z|X}(z|x) f_X(x) dx =& \frac{1}{4}\left(\mathcal{N}(2, \sigma_N^2)|_{x=z} + \mathcal{N}(-2, \sigma_N^2)|_{x=z}\right) \\
\Rightarrow \int_{-\infty}^{\hat{x}_{MAE}} \mathcal{N}(x, \sigma_N^2)(\delta(x-2) + \delta(x+2)) dx =& \frac{1}{2}\left(\mathcal{N}(2, \sigma_N^2)|_{x=z} + \mathcal{N}(-2, \sigma_N^2)|_{x=z}\right)
\end{aligned}
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   Note the LHS is not continuous $\Rightarrow \hat{x}_{MAE}$ not well defined.

   (b) Note $f_Z(z)$ is symmetric about 0 $\Rightarrow E\{\hat{x}_{ML}\} = E\{z\} = 0 \Rightarrow \hat{x}_{ML}$ is unbiased ($E\{x\} = 0$). Similarly, $E\{\hat{x}_{MAP}\} = 2Pr\{z > 0\} - 2Pr\{z < 0\} = 0 \Rightarrow \hat{x}_{MAP}$ is unbiased. Also, $\hat{x}_{MS}$ is an odd function (about 0) of $z \Rightarrow E\{\hat{x}_{MS}\} = 0 \Rightarrow \hat{x}_{MS}$ is unbiased.

   (c) $\sigma_{ML}^2 = \sigma_Z^2 = \sigma_X^2 + \sigma_N^2 = 4 + \sigma_N^2$. Also, $\sigma_{MAP}^2 = 4$ (since $\hat{x}_{MAP} = \pm 2$). Determining σ_{MS}^2 is not trivial, and will not be considered.