1. (30 pts) Probability questions:

- (15 pts) Let x be a random variables with density $f_x(x)$ given below. Let $y = g(x)$ be the shown function. Determine $f_y(y)$ and $F_y(y)$.

- (15 pts) Let x and y be independent, zero mean, unit variance Gaussian random variables. Define

$$w = x^2 + y^2 \qquad \text{and} \qquad z = x^2.$$

Determine $f_{w,z}(w,z)$. Are w and z independent?

Answer: Note that

$$f_x(x) = \begin{cases} \frac{1}{4}x + \frac{1}{2}\delta(x-0.5) & 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$F_x(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8}x^2 + \frac{1}{2}u(x-0.5) & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

Since $x = \sqrt{y}$ for $0 \ge y \ge 1$,

$$\begin{aligned} F_y(y) &= \begin{cases} 0 & y < 0 \\ F_x(\sqrt{y}) & 0 \le y < 1 \\ 1 & 1 \le y \end{cases} \\ &= \begin{cases} 0 & y < 0 \\ \frac{1}{8}y + \frac{1}{2}u(\sqrt{y} - 0.5) & 0 \le y < 1 \\ 1 & 1 \le y \end{cases} \\ &= \begin{cases} 0 & y < 0 \\ \frac{1}{8}y + \frac{1}{2}u(y - 0.25) & 0 \le y < 1 \\ 1 & 1 \le y \end{cases} \end{aligned}$$

Taking the derivitive yields

$$f_y(y) = \begin{cases} \frac{1}{8} + \frac{1}{2}\delta(y-0.25) + \frac{3}{8}\delta(y-1) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Tha Jabobian of the transformation is

$$J(x,y) = \left| \begin{array}{cc} \frac{d(x^2+y^2)}{dx} & \frac{d(x^2+y^2)}{dy} \\ \frac{d(x^2)}{dx} & \frac{d(x^2)}{dy} \end{array} \right| = \left| \begin{array}{cc} 2x & 2y \\ 2x & 0 \end{array} \right| = 4|xy|$$

The reverse transformation is easily seen to be $x = \pm\sqrt{z}$ and $y = \pm\sqrt{w - x^2} = \pm\sqrt{w-z}$, $w \geq z$. Thus,

$$\begin{aligned} f_{w,z}(w,z) \quad = \quad & \left. \frac{f_{x,y}(x,y)}{4|xy|} \right|_{\substack{x=\sqrt{z} \\ y=\sqrt{w-z}}} + \left. \frac{f_{x,y}(x,y)}{4|xy|} \right|_{\substack{x=\sqrt{z} \\ y=-\sqrt{w-z}}} \\ & + \left. \frac{f_{x,y}(x,y)}{4|xy|} \right|_{\substack{x=-\sqrt{z} \\ y=\sqrt{w-z}}} + \left. \frac{f_{x,y}(x,y)}{4|xy|} \right|_{\substack{x=-\sqrt{z} \\ y=-\sqrt{w-z}}} \end{aligned} \tag{1}$$

Since x and y are independent,

$$f_{x,y}(x,y) = \frac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}}$$

Thus

$$f_{w,z}(w,z) = \frac{1}{2\pi\sqrt{z}\sqrt{w-z}} e^{-w/2} u(w) u(z) u(w-z)$$

where the last three terms indicate $w, z \geq 0$ and $w \geq z$.

2. (30 pts) Random processes: Consider the ARMA(1,1) process defined by:

$$x[n] = -a_1 x[n-1] + b_0 v[n] + b_1 v[n-1]$$

where $v[n]$ is zero mean white Gaussian process with variance σ_v^2.

- (10 pts) Determine the mean of the process.
- (20 pts) Prove that the autocorrelation is given by

$$r[k] = \begin{cases} \frac{(b_0^2+b_1^2-2a_1b_0b_1)\sigma_v^2}{1-a_1^2} & k=0 \\ \frac{(a_1^2b_0b_1-a_1b_0^2-a_1b_1^2+b_0b_1)\sigma_v^2}{1-a_1^2} & k=1 \\ (-a_1)^{k-1}r[1] & k\geq 2 \end{cases}$$

Answer: Consider first the mean:

$$\begin{aligned} E[x[n]] &= E[-a_1x[n-1] + b_0v[n] + b_1v[n-1]] \\ \mu_x &= -a_1\mu_x + 0 + 0 \\ &= 0 \end{aligned}$$

Now evaluate $r[0]$,

$$\begin{aligned} r[0] &= E[x[n](-a_1x[n-1] + b_0v[n] + b_1v[n-1])] \\ &= -a_1E[x[n]x[n-1]] + b_0E[x[n]v[n]] + b_1E[x[n]v[n-1]] \end{aligned}$$

Evaluate the last two terms separately:

$$\begin{aligned} E[x[n]v[n]] &= E[(-a_1x[n-1] + b_0v[n] + b_1v[n-1])v[n]] \\ &= 0 + b_0\sigma_v^2 + 0 = b_0\sigma_v^2 \end{aligned}$$

and

$$\begin{aligned} E[x[n]v[n-1]] &= E[(-a_1x[n-1] + b_0v[n] + b_1v[n-1])v[n-1]] \\ &= -a_1E[x[n-1]v[n-1]] + 0 + b_1\sigma_v^2 \\ &= -a_1b_0\sigma_v^2 + b_1\sigma_v^2 \end{aligned}$$

where the last line uses the fact that $E[x[n-1]v[n-1]] = E[x[n]v[n]]$. Substituting these terms in the expression for $r[0]$,

$$r[0] = -a_1r[1] + b_0\sigma_v^2 - a_1b_0\sigma_v^2 + b_1\sigma_v^2$$

Now evaluate $r[1]$,

$$\begin{aligned} r[1] &= E[x[n-1](-a_1x[n-1] + b_0v[n] + b_1v[n-1])] \\ &= -a_1E[x[n-1]x[n-1]] + b_0E[x[n-1]v[n]] + b_1E[x[n-1]v[n-1]] \end{aligned}$$

Note that $E[x[n-1]v[n]] = 0$ and

$$E[x[n-1]v[n-1]] = E[x[n]v[n]] = b_0\sigma_v^2$$

Thus we have

$$\begin{aligned} r[0] &= -a_1 r[1] + b_0\sigma_v^2 - a_1 b_0 \sigma_v^2 + b_1\sigma_v^2 \\ r[1] &= -a_1 r[0] + b_0\sigma_v^2 + b_0 b_1 \sigma_v^2 \end{aligned}$$

Solving for $r[0]$ and $r[1]$ gives the desired result for $k = 0, 1$. Note for $k \geq 2$,

$$\begin{aligned} r[k] &= E[x[n-k](-a_1 x[n-1] + b_0 v[n] + b_1 v[n-1])] \\ &= -a_1 E[x[n-k]x[n-1]] + b_0 E[x[n-k]v[n]] + b_1 E[x[n-k]v[n-1]] \\ &= -a_1 r[k-1] + 0 + 0 \\ &= (-a_1)^{k-1} r[1] \end{aligned}$$

which completes the proof.

3. (40 pts) Estimation:

- (20 pts) Let x and a be random variables. The conditional probability density function for x and the probability density function for a are given by

$$f_{x|a}(x|a) = ae^{-ax}u(x) \quad \text{and} \quad f_a(a) = \begin{cases} \frac{1}{2} & 1 \le a \le 3 \\ 0 & \text{otherwise} \end{cases} .$$

  Assuming you observe x, determine the ML and MAP estimates of a. Explain any similarities or differences in the estimates.

- (20 pts) Let x and y be random variables with joint density

$$f_{x,y}(x,y) = \begin{cases} 8xy & 0 \le x \le 1, 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

  Given that you observe x, determine the Bayes estimate of y for the squared, absolute, and uniform error cases.

Answer: The ML estimate maximizes $f_{x|a}(x|a) = ae^{-ax}u(x)$. Differentiating and equating to 0,

$$-\hat{a}_{ML}xe^{-\hat{a}_{ML}x} + e^{-\hat{a}_{ML}x} = 0$$

or $\hat{a}_{ML} = \frac{1}{x}$. In the MAP case, we must maximize

$$\begin{aligned} f_{a|x}(a|x) &= \frac{f_{x|a}(x|a)f_a(a)}{f_x(x)} \\ &= \frac{ae^{-ax}u(x)(u(a-1) - u(a-3))}{f_x(x)} \end{aligned}$$

Note that we need only need maximize the numerator, which is identical to the ML estimate for $1 \le \hat{a} \le 3$. If $\frac{1}{x}$ is outside this range, we must check the boundary conditions, which gives

$$\hat{a}_{MA} = \begin{cases} 1 & 0 < \frac{1}{x} \le 1 \\ \frac{1}{x} & 1 < \frac{1}{x} \le 3 \\ 3 & 3 < \frac{1}{x} \end{cases}$$

Note that in the range [1,3] the ML and MAP estimates are identical as the *a priori* distribution of a yields no information (flat). Outside that range this information does provide a benefit.

For the second part, integrating to get the marginals gives

$$f_x(x) = \int_0^x 8xydy = 4xy^2|_0^x = 4x^3, 0 \le x \le 1$$

and

$$f_{y|x}(y|x) = \frac{f_{x,y}(x,y)}{f_x(x)} = \begin{cases} \frac{2y}{x^2} & 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Maximizing this gives $\hat{y}_{MAP} = x$. In the SE case, $\hat{y}_{SE} = E[y|x] = \int_0^x \frac{2y^2}{x^2} = \frac{2x}{3}$. Also,

$$F_{y|x}(y|x) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{x^2} & 0 \leq y \leq x \\ 1 & x < y \end{cases}$$

For the AE estimate, $F_{y|x}(\hat{y}_{AE}|x) = 1/2$, or $\hat{y}_{AE} = \frac{x}{\sqrt{2}}$.