1. (35 pts) Probability questions:

- (10 pts) Let x be a random variable and set $y = x^2$. Derive a simplified expression for $f(y|x \geq 0)$.
- (15 pts) Suppose now that $y = a\sin(x+\theta)$, where θ and $a > 0$ are constants. Determine $f_y(y)$.
- (10 pts) Suppose further that x is uniformly distributed over $[-\pi, \pi]$. Determine $f_y(y)$ for this special case.

Answer: Clearly, $F(y|x \geq 0) = 0$ for $y < 0$. Then for $y \geq 0$,

$$F(y|x \geq 0) = \frac{Pr(Y \leq y, X \geq 0)}{Pr(X \geq 0)} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)} U(y).$$

Thus

$$f(y|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}(1 - F_x(0))} U(y).$$

Now for $y = g(x) = a\sin(x+\theta)$ we have, assuming $|y| \leq a$, infinitely many solutions

$$x_n = \arcsin(y/a) - \theta$$

$n = 0, \pm 1, \pm 2, \ldots$. Also,

$$g'(x_n) = a\cos(x_n + \theta)$$

Note that $g^2(x_n) + g'^2(x_n) = a^2\cos^2(x_n+\theta) + a^2\sin^2(x_n+\theta) = a^2$. Or,

$$g'(x_n) = \sqrt{a^2 - g^2(x_n)} = \sqrt{a^2 - y^2}.$$

Thus

$$f_y(y) = \sum_i \frac{f_x(x_n)}{g'(x_n)} = \frac{1}{\sqrt{a^2 - y^2}} \sum_i f_x(x_n), \quad |y| \leq a$$

If $x \sim U(-\pi, \pi)$ then there is only a single solution, and

$$f_y(y) = \frac{1}{2\pi\sqrt{a^2 - y^2}}, \quad |y| \leq a$$

2. (35 pts) Suppose a real sequence is generated according to the model

$$x(n) = w_0 x^2(n-1) + w_1 x(n-1) + w_2 x(n-1)x(n-2) + w_3 x(n-2) + w_4 x^2(n-2) + v(n)$$

where $v(n)$ is a white noise process with variance σ_v^2.

- (30 pts) Determine a set of Yule–Walker type equations for determining w_0, w_1, w_2, w_3, w_4 and σ_v^2.
- (5 pts) Based on this result, what statistics are needed to characterize signals generated by the general model

$$x(n) = \sum_{i=1}^{M} w_i x(n-i) + \sum_{i=1}^{M}\sum_{j=1}^{M} w_{i,j} x(n-i)x(n-j) + v(n)$$

Answer: Let $r(k) = E[x(n)x(n-k)]$ and $r(k,l) = E[x(n)x(n-k)x(n-l)]$. Assume first $k < 0$,

$$\begin{aligned}
E[x(n)x(n-k)] &= w_0 E[x^2(n-1)x(n-k)] + w_1 E[x(n-1)x(n-k)] + \\
&\quad w_2 E[x(n-1)x(n-2)x(n-k)] + w_3 E[x(n-2)x(n-k)] \\
&\quad + w_4 E[x^2(n-2)x(n-k)] + E[x(n-k)v(n)] \\
r(k) &= w_0 r(0,k-1) + w_1 r(k-1) + w_2 r(1,k-1) + w_3 r(k-2) + w_4 r(0,k-2)
\end{aligned}$$

Or letting $k = 1,2,3,4,5$ and writing the results in matrix form

$$\begin{bmatrix} r(0,0) & r(0) & r(1,0) & r(1) & r(0,1) \\ r(0,1) & r(1) & r(1,1) & r(0) & r(0,0) \\ r(0,2) & r(2) & r(1,2) & r(1) & r(0,1) \\ r(0,3) & r(3) & r(1,3) & r(2) & r(0,2) \\ r(0,4) & r(4) & r(1,4) & r(3) & r(0,3) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_1 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \\ r(3) \\ r(4) \\ r(5) \end{bmatrix}$$

Letting $k = 0$,

$$\begin{aligned}
r(0) &= w_0 r(0,1) + w_1 r(1) + w_2 r(1,1) + w_3 r(2) + w_4 r(0,2) + E[x(n)v(n)] \\
&= w_0 r(0,1) + w_1 r(1) + w_2 r(1,1) + w_3 r(2) + w_4 r(0,2) + \\
&\quad w_0 E[x^2(n-1)v(n)] + w_1 E[x(n-1)v(n)] + \\
&\quad w_2 E[x(n-1)x(n-2)v(n)] + w_3 E[x(n-2)v(n)] \\
&\quad + w_4 E[x^2(n-2)v(n)] + E[v(n)v(n)] \\
&= w_0 r(0,1) + w_1 r(1) + w_2 r(1,1) + w_3 r(2) + w_4 r(0,2) + \sigma_v^2
\end{aligned}$$

Or,

$$\sigma_v^2 = r(0) - w_0 r(0,1) - w_1 r(1) - w_2 r(1,1) - w_3 r(2) - w_4 r(0,2)$$

In general, polynomial systems are determined by the set of higher order statistics $r(k) = E[x(n)x(n-k)]$ and $r(k,l) = E[x(n)x(n-k)x(n-l)]$.

3. (30 pts) Let $s(n)$ and $w(n)$ be independent random variables, where $s(n)$ has correlation $r_s(l) = p^{|l|}$ and $w(n)$ is independent of $s(n)$ and is white with variance σ^2.

- (20 pts) Let an observed sequence be generated by

$$x(n) = s(n) + w(n).$$

  Suppose we wish to estimate $s(n)$ from $x(n-k)$ as

$$\hat{s}(n) = a_k x(n-k).$$

  using mean squared error as the cost function. Directly utilize the MSE cost function to determine the MSE optimal coefficient a_k, i.e., minimize $E[(s(n) - \hat{s}(n))^2]$.

- (10 pts) What is the optimal coefficient if the noise is multiplicative,

$$x(n) = s(n)w(n).$$

  Explain the result.

Answer:

$$\begin{aligned} E[(s(n) - \hat{s}(n))^2] &= E[(s(n) - a_k x(n-k))^2] \\ &= E[(s^2(n)] - 2a_k E[s(n)x(n-k)] + a_k^2 E[x^2(n-k)] \\ &= E[(s^2(n)] - 2a_k E[s(n)(s(n-k) + w(n-k))] + a_k^2 E[x^2(n-k)] \\ &= \sigma_s^2 - 2a_k r(k) + a_k^2 \sigma_x^2 \end{aligned}$$

Differentiating and equating to 0,

$$a_k = \frac{r(k)}{\sigma_x^2} = \frac{p^{|k|}}{1 + \sigma^2}$$

In the multiplicative case,

$$\begin{aligned} E[(s(n) - \hat{s}(n))^2] &= E[(s(n) - a_k x(n-k))^2] \\ &= E[(s^2(n)] - 2a_k E[s(n)x(n-k)] + a_k^2 E[x^2(n-k)] \\ &= E[(s^2(n)] - 2a_k E[s(n)s(n-k)w(n-k)] + a_k^2 E[x^2(n-k)] \\ &= \sigma_s^2 - 0 + a_k^2 \sigma_x^2 \end{aligned}$$

Differentiating and equating to 0,

$$a_k = 0.$$

Reason: the multiplicative noise makes the desired and observed sequences uncorrelated.