1. (30 pts) The random variables x and y are independent and uniformly distributed on the interval [0,1]. Determine the conditional distribution $f_{r|A}(r|A)$ where $r = \sqrt{x^2+y^2}$ and $A = \{r \leq 1\}$.

Answer:

Examine the joint density $f_{x,y}(x,y)$ in the xy plane. Since x and y are independent,

$$f_{x,y}(x,y) = f_x(x)f_y(y) = 1 \quad \text{for } 0 \leq x, y \leq 1$$

This defines a uniform density over the region $0 \leq x, y \leq 1$ in the first quadrant of the xy plane. Note that $r = \sqrt{x^2+y^2}$ defines an arc in the first quadrant. Also, if $0 \leq r \leq 1$ the area under the uniform density up to radius r is simply given by

$$\begin{aligned} F_r(r) &= Pr[\sqrt{x^2+y^2} \leq r] = \int_{\sqrt{x^2+y^2}\leq r} f_{x,y}(x,y)dxdy \\ &= \int_{\sqrt{x^2+y^2}\leq r} 1dxdy = \frac{\pi r^2}{4} \quad \text{for } 0 \leq r \leq 1 \end{aligned}$$

Then for $A = \{r \leq 1\}$.

$$F_{r|A}(r|A) = \frac{F_{r,A}(r,A)}{Pr[A]} = \frac{F_r(r)}{F_r(1)} = \frac{\pi r^2}{4}\frac{4}{\pi} = r^2 \quad \text{for } 0 \leq r \leq 1$$

Thus, $f_{r|A}(r|A) = 2r$ for $0 \leq r \leq 1$ and 0 elsewhere.

2. (35 pts) A random process is defined by

$$x[n] = s[n] + \eta[n]$$

where $\eta[n]$ is a unit variance white noise process and $s[n]$ is given by

$$s[n] = \rho s[n-1] + w[n]$$

and $w[n]$ is also a unit variance white noise process independent of $\eta[n]$.

- (20 pts) What is the correlation function $R_x[l]$ and the PSD $S_x(\omega)$?
- (15 pts) Suppose $\rho = 0.5$ and $x[n]$ is to be modeled as a second order AR process. Determine the optimal values of all parameters necessary to describe the second order AR model.

Answer:

Consider first the correlation of $s[n]$. Let the lag be $l = 0$,

$$\begin{aligned} r_s[0] &= E[s[n]s[n]] = E[(\rho s[n-1] + w[n])(\rho s[n-1] + w[n])] \\ &= \rho^2 E[s[n-1]s[n-1]] + 2E[s[n-1]w[n]) + E[w[n]w[n]] \\ &= \rho^2 r_s[0] + 1 = \frac{1}{1+\rho^2} \end{aligned}$$

Recall that the correlation function has the same characteristic function as the time domain signal. Thus for $l > 0$,

$$r_s[l] = \rho r_s[l-1]$$

or combining with the results for $r_s[0]$,

$$r_s[l] = \frac{\rho^{|l|}}{1+\rho^2}.$$

And since $s[n]$ and $\eta[n]$ are independent, their correlation functions add, giving

$$r_x[l] = \delta[l] + \frac{\rho^{|l|}}{1+\rho^2}.$$

Taking the Fourier transform

$$\begin{aligned} R_x[z] &= 1 + \frac{1}{1+\rho^2}\left(1 + \sum_{l=1}^{\infty} \rho^l z^l + \rho^l z^{-l}\right) \\ &= 1 + \frac{1}{1+\rho^2}\left(-1 + \frac{1}{1-\rho z} + \frac{1}{1-\rho z^{-1}}\right) \\ &= 1 + \frac{1}{1+\rho^2}\left(-1 + \frac{2-\rho(z+z^{-1})}{1+\rho^2-\rho(z+z^-1)}\right) \\ &= 1 + \frac{1}{1+\rho^2}\left(\frac{1-\rho^2}{1+\rho^2-\rho(z+z^-1)}\right) \\ &= 1 + \frac{1-\rho^2}{1+\rho^2}\left(\frac{1}{1+\rho^2-\rho(z+z^-1)}\right) \end{aligned}$$

Evaluating on the unit circle, $z = e^{j\omega}$, gives the FT and the desired PSD

$$S_x(\omega) = 1 + \frac{1-\rho^2}{1+\rho^2}\left(\frac{1}{1+\rho^2 - 2\rho\cos\omega}\right)$$

For $\rho = 0.5$ and $N = 2$,

$$\mathbf{R} = \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} = \begin{bmatrix} 1.8 & 1.4 \\ 1.4 & 1.8 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.2 \end{bmatrix}$$

Thus

$$\mathbf{w} = \mathbf{R}^{-1}\mathbf{r} = \begin{bmatrix} 0.6562 \\ 0.1563 \end{bmatrix}$$

and the AR model is

$$x[n] = 0.6562x[n-1] + 0.1563x[n-2] + v[n]$$

Where $v[n]$ is WGN with

$$\sigma_v^2 = \sum_{i=0}^{N} a_i r(i) = 1.8 - 0.6562 \times 1.4 - 0.1563 \times 1.2 = 0.6937$$

3. (35 pts) The time between new cell phone connections to a local base station is governed by the exponential distribution

$$f_T(T) = \begin{cases} \alpha e^{-\alpha T} & T \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

- (15 pts) Derive the ML estimate of α assuming N independent observations, $T_1, T_2, \ldots, T_N$, are recorded.

Suppose now that α is a RV described by the distribution

$$f_\alpha(\alpha) = \begin{cases} \alpha\beta^2 e^{-\alpha\beta} & \alpha \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

- (10 pts) Derive the MAP estimate of α.
- (10 pts) Derive the mean–square estimate of α.

Note: $\int_0^\infty y^n e^{-y} dy = n!$

Answer:

$$f_{\mathbf{T}|\alpha}(\mathbf{T}|\alpha) = \alpha^N e^{-\alpha \sum_{i=1}^N T_i}$$

Taking the log, differentiating, and equating to 0 yields

$$\alpha_{ML} = \frac{1}{\frac{1}{N}\sum_{i=1}^N T_i}$$

For α a RV,

$$f_{\alpha|\mathbf{T}}(\alpha|\mathbf{T}) = \frac{f_{\mathbf{T}|\alpha}(\mathbf{T}|\alpha) f_\alpha(\alpha)}{f_{\mathbf{T}}(\mathbf{T})} = \frac{1}{f_{\mathbf{T}}(\mathbf{T})}\alpha^{N+1}\beta^2 e^{-\alpha(\beta+\sum_{i=1}^N T_i)}$$

Ignoring the denominator, which is not a function of α, taking the log, differentiating, and equating to 0 yields

$$\alpha_{MAP} = \frac{1}{\frac{1}{N+1}(\beta + \sum_{i=1}^N T_i)}$$

The MSE estimate is given by

$$\begin{aligned} \alpha_{MSE} = E[\alpha|\mathbf{T}] &= \int_0^\infty \alpha f_{\alpha|\mathbf{T}}(\alpha|\mathbf{T}) d\alpha = \int_0^\infty \frac{\alpha f_{\mathbf{T}|\alpha}(\mathbf{T}|\alpha) f_\alpha(\alpha)}{f_{\mathbf{T}}(\mathbf{T})} d\alpha \\ &= \frac{1}{f_{\mathbf{T}}(\mathbf{T})} \int_0^\infty \alpha^{N+2}\beta^2 e^{-\alpha(\beta+\sum_{i=1}^N T_i)} d\alpha \end{aligned}$$

Letting $\phi = \beta + \sum_{i=1}^N T_i$ and $\theta = \alpha\phi$

$$\begin{aligned} \alpha_{MSE} &= \frac{\beta^2}{f_{\mathbf{T}}(\mathbf{T})} \int_0^\infty \alpha^{N+2} e^{-\alpha\phi} d\alpha = \frac{\beta^2 \phi^{-(N+3)}}{f_{\mathbf{T}}(\mathbf{T})} \int_0^\infty \theta^{N+2} e^{-\theta} d\theta = \frac{\beta^2 (N+2)! \phi^{-(N+3)}}{f_{\mathbf{T}}(\mathbf{T})} \\ &= \frac{\beta^2 (N+2)!}{f_{\mathbf{T}}(\mathbf{T})(\beta + \sum_{i=1}^N T_i)^{N+3}} \end{aligned}$$

Similar arguments show that

$$f_{\mathbf{T}}(\mathbf{T}) = \int_0^\infty \alpha^{N+1}\beta^2 e^{-\alpha(\beta+\sum_{i=1}^N T_i)}d\alpha = \frac{\beta^2(N+1)!}{(\beta+\sum_{i=1}^N T_i)^{N+2}}$$

Thus,

$$\alpha_{MSE} = \frac{\beta^2(N+2)!}{f_{\mathbf{T}}(\mathbf{T})(\beta+\sum_{i=1}^N T_i)^{N+3}} = \frac{N+2}{\beta+\sum_{i=1}^N T_i} = \frac{1}{\frac{1}{N+2}(\beta+\sum_{i=1}^N T_i)}$$