1. (35 pts) Let $y = \min\{|x_1|, x_2\}$ where x_1 and x_2 are i.i.d. inputs with cdf and pdf $F_x(\cdot)$ and $f_x(\cdot)$, respectively. For simplicity, assume $f_x(\cdot)$ is symmetric about 0, i.e., $f_x(x) = f_x(-x)$. Determine the cdf and pdf of y in terms of the distribution of the inputs. Plot the pdf of y for $f_x(\cdot)$ uniform on $[-1, 1]$.

Note that

$$F_{|x|}(x) = \begin{cases} F_x(x) - F_x(-x) & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Also

$$F_{\min\{x_1,x_2\}}(x) = 1 - P\{x_1 \geq x\}P\{x_2 \geq x\} = 1 - (1 - F_{x_1}(x))(1 - F_{x_2}(x))$$

Thus,

$$F_y(y) = 1 - (1 - F_{|x_1|}(y))(1 - F_{x_2}(y))$$

$$= \begin{cases} 1 - (1 - F_x(y) + F_x(-y))(1 - F_x(y)) & \text{for } y \geq 0 \\ 1 - (1 - F_{x_2}(y)) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2F_x(y) - F_x(-y) - F_x^2(y) + F_x(y)F_x(-y) & \text{for } y \geq 0 \\ F_x(y) & \text{otherwise} \end{cases}$$

If $f_x(\cdot)$ is symmetric about 0, then $f_x(x) = f_x(-x)$ and $F_x(x) = 1 - F_x(-x)$, giving

$$F_y(y) = \begin{cases} 2F_x(y) - (1 - F_x(y)) - F_x^2(y) + F_x(y)(1 - F_x(y)) & \text{for } y \geq 0 \\ F_x(y) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 4F_x(y) - 2F_x^2(y) - 1 & \text{for } y \geq 0 \\ F_x(y) & \text{otherwise} \end{cases}$$

Taking the derivative,

$$f_y(y) = \begin{cases} 4f_x(y) - 4f_x(y)F_x(y) & \text{for } y \geq 0 \\ f_x(y) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 4f_x(y)(1 - F_x(y)) & \text{for } y \geq 0 \\ f_x(y) & \text{otherwise} \end{cases}$$

2. (35 pts) Consider the observed samples

$$y_i = \theta + x_i$$

for $i = 1, 2, \ldots, N$. We wish to estimate the location parameter θ using a maximum likelihood estimator operating on the observations $y_1, y_2, \ldots, y_N$. Consider two cases:

- (10 pts) The x_i terms are i.i.d. with distribution $x_i \sim \mathcal{N}(0, \sigma^2)$, for $i = 1, 2, \ldots, N$.
- (10 pts) The x_i terms are independent with distribution $x_i \sim \mathcal{N}(0, \sigma_i^2)$, for $i = 1, 2, \ldots, N$.
- (15 pts) Are the estimates unbiased? What is the variance of the estimates? Are they consistent?

$$f_{\mathbf{y}|\theta}(\mathbf{y}|\theta) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-\theta)^2}{2\sigma^2}} = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\sum_{i=1}^{N} \frac{(y_i-\theta)^2}{2\sigma^2}}$$

Thus,

$$\theta_{ML} = \arg\max_{\theta} - \sum_{i=1}^{N} \frac{(y_i-\theta)^2}{2\sigma^2}$$

and taking the derivative,

$$\sum_{i=1}^{N} \frac{(y_i - \theta_{ML})}{\sigma^2} = 0 \Rightarrow \theta_{ML} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

For the case of changing variances,

$$\sum_{i=1}^{N} \frac{(y_i - \theta_{ML})}{\sigma_i^2} = 0 \Rightarrow \theta_{ML} = \frac{\sum_{i=1}^{N} \frac{y_i}{\sigma_i^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}} \theta_{ML} = \frac{\sum_{i=1}^{N} w_i y_i}{\sum_{i=1}^{N} w_i}$$

which is a normalized filter, where $w_i = \frac{1}{\sigma^2}$ for $i = 1, 2, \ldots, N$.

For each estimate $E\{\theta_{ML}\} = \theta$, and they are thus unbiased.

$$\text{var}(\theta_{ML})[N] = E\{(\theta_{ML} - \theta)^2\} = E\left\{\left(\frac{\sum_{i=1}^{N} w_i y_i - w_i\theta}{\sum_{i=1}^{N} w_i}\right)^2\right\} = E\left\{\left(\frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i}\right)^2\right\}$$

$$= \frac{E\{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i x_i x_j w_j\}}{(\sum_{i=1}^{N} w_i)^2} = \frac{\sum_{i=1}^{N} w_i^2 \sigma_i^2}{(\sum_{i=1}^{N} w_i)^2} = \frac{\sum_{i=1}^{N} w_i}{(\sum_{i=1}^{N} w_i)^2} = \frac{1}{\sum_{i=1}^{N} w_i}$$

Since $w_i > 0$, we have $\text{var}(\theta_{ML})[N+1] < \text{var}(\theta_{ML})[N]$. This, combined with the fact that the estimator is unbiased means the estimate is consistent.

3. (30 pts) Noncausal IIR Weiner filter design: Suppose a noncausal filter with infinite impulse response estimates a desired signal, $\{d(n)\}$, based on the observed signal $\{x(n)\}$. The estimate in this case is given by

$$\hat{d}(n) = \sum_{l=-\infty}^{\infty} x(l)h(n-l)$$

where $\{h(n)\}$ is the infinite, noncausal filter impulse response and where all signals are assumed to be WSS. Utilizing the orthogonality principle, show that the optimal noncausal, IIR Weiner filter is given by

$$H(\omega) = \frac{S_{dx}(\omega)}{S_x(\omega)}.$$

From the orthogonality property,

$$E\{x(n-k)e^*(n)\} = E\left\{x(n-k)\left(d^*(n) - \sum_{l=-\infty}^{\infty} x^*(n-l)h^*(l)\right)\right\} = 0$$

$$E\{x(n-k)d^*(n)\} = \sum_{l=-\infty}^{\infty} E\{x(n-k)x^*(n-l)\}h^*(l)$$

$$r_{xd}(-k) = \sum_{l=-\infty}^{\infty} r_x(l-k)h^*(l) = \sum_{l=-\infty}^{\infty} r_x^*(k-l)h^*(l) = r_x^*(k) \otimes h^*(k)$$

Thus taking the complex conjugate,

$$r_{xd}^*(-k) = r_{dx}(k) = r_x(k) \otimes h(k)$$

and taking the Fourier transform,

$$S_{dx}(\omega) = S_x(\omega)H(\omega)$$

or

$$H(\omega) = \frac{S_{dx}(\omega)}{S_x(\omega)}$$