1. Let

$$\mathbf{R} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

Express $\mathbf{R}$ as $\mathbf{R} = \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^H$, where $\boldsymbol{\Omega}$ is diagonal.

Solution:

$$\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = 6, \lambda_2 = 1$$

Than solving $\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i$ gives $\mathbf{q}_1 = \frac{1}{\sqrt{5}}[1, -2]^T$ and $\mathbf{q}_2 = \frac{1}{\sqrt{5}}[2, 1]^T$. Thus $\mathbf{R} = \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^H$ where

$$\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2] \qquad \text{and} \qquad \boldsymbol{\Omega} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

2. The two-dimensional covariance matrix can be expressed as:

$$\mathbf{C} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho^*\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

(a) Find the simplest expression for the eigenvalues of $\mathbf{C}$.

(b) Specialize the results to the case $\sigma^2 = \sigma_2^2 = \sigma_2^2$.

(c) What are the eigenvectors in the special case (b) when ρ is real?

Solution:

(a)

$$\begin{vmatrix} \sigma_1^2 - \lambda & \rho\sigma_1\sigma_2 \\ \rho^*\sigma_1\sigma_2 & \sigma_2^2 - \lambda \end{vmatrix} = \lambda^2 - (\sigma_1^2 + \sigma_2^2)\lambda + (1 - |p|^2)\sigma_1^2\sigma_2^2 = 0$$

$$\Rightarrow \lambda = \frac{(\sigma_1^2 + \sigma_2^2) \pm \sqrt{\sigma_1^4 + \sigma_2^4 - 2\sigma_1^2\sigma_2^2 + 4|p|^2\sigma_1^2\sigma_2^2}}{2}$$

(b) For $\sigma^2 = \sigma_2^2 = \sigma_2^2$

$$\lambda = \frac{2\sigma^2 \pm \sqrt{4|p|^2\sigma^4}}{2} = (1 \pm |p|)\sigma^2$$

3. Let

$$x[n] = Ae^{j\omega_0 n}$$

where the complex amplitude A is a RV with random magnitude and phase

$$A = |A|e^{j\phi}.$$

Show that a sufficient condition for the random process to be stationary is that the amplitude and phase are independent and that the phase is uniformly distributed over $[-\pi, \pi]$.

Solution: First note $E\{x[n]\} = E\{A\}e^{j\omega_0 n}$ and

$$E\{A\} = E\{|A|\}E\{e^{j\phi}\} = 0$$

by independence and uniform distribution of ϕ. Thus it has a fixed mean. Next note

$$E\{x[n]x^*[n-k]\} = E\{|A|^2\}e^{j\omega_0 k}$$

which is strictly a function of $k \Rightarrow$ WSS.

4. Let X_i be i.i.d. RVs uniformly distributed on $[0, 1]$ and define

$$Y = \sum_{i=1}^{20} X_i.$$

Utilize Tchebycheff's inequality to determine a bound for $Pr\{8 < Y < 12\}$.

Solution: Note $\eta_x = \frac{1}{2}$ and $\sigma_x^2 = \frac{1}{12}$. Thus $\eta_y = 10$ and $\sigma_y^2 = \frac{20}{12} = \frac{5}{3}$. Utilize Tchebycheff's inequality

$$Pr\{|Y - \eta_y| \geq 2\} \leq \left(\frac{\sigma_y}{2}\right)^2 = \frac{5}{12} \qquad \Rightarrow \qquad Pr\{8 < Y < 12\} \geq 1 - \frac{5}{12} = \frac{7}{12}$$

5. Let $X \sim \mathcal{N}(0, 2\sigma^2)$ and $Y \sim \mathcal{N}(1, \sigma^2)$ be independent RVs. Also, define $Z = XY$. Find the Bays estimate of X from observation Z:

   (a) Using the squared error criteria.

   (b) Using the absolute error criteria.

6. Let X and Y be independent RVs characterized by $f_X(x) = ae^{-ax}U(x)$ and $f_Y(y) = ae^{-ay}U(y)$. Also, define $Z = XY$. Find the Bays estimate of X from observation Z using the uniform cost function.

Solution:

$$F_{z|x}(z|x) = Pr(xy \leq z|x) = Pr(y \leq z/x) = F_y(z/x) \qquad \Rightarrow \qquad f_{z|x}(z|x) = \frac{1}{x}f_y(z/x)$$

$$\begin{aligned}
\hat{x} =& \arg\max f_{z|x}(z|x)f_x(x) = \arg\max \frac{1}{x}f_y(z/x)f_x(x) \\
=& \arg\max \frac{1}{x}ae^{-az/x}ae^{-ax}U(x)U(z) = \arg\max a^2x^{-1}e^{-a(zx^{-1}+x)}U(x)U(z) \\
\Rightarrow 0 =& -a^2x^{-2}e^{-a(zx^{-1}+x)} + (a^2x^{-1}e^{-a(zx^{-1}+x)})(-a(1-zx^{-2})) \\
0 =& -x^{-1} - a(1-zx^{-2}) \Rightarrow ax^2 + x - z = 0 \\
\Rightarrow \hat{x} =& \frac{-1 \pm \sqrt{1+4az}}{2a}
\end{aligned}$$

7. Random processes $x[n]$ and $y[n]$ are defined by

$$\begin{aligned}
x[n] &= v_1[n] + 3v_2[n-1] \\
y[n] &= v_2[n+1] + 3v_2[n-1]
\end{aligned}$$

where $v_1[n]$ and $v_2[n]$ are independent white noise processes, each with variance 0.5.

(a) Determine the autocorrelation functions of x and y. Are the processes WSS?

(b) Determine the cross–correlation functions between x and y. Are the processes jointly WSS?

Solution:

(a)

$$\begin{aligned} r_{xx}(n, n-k) =& E\{(v_1[n] + 3v_2[n-1])(v_1[n-k] + 3v_2[n-1-k])^*\} \\ =& \sigma^2\delta(k) + 9\sigma^2\delta(k) = 5\delta(k) \\ r_{yy}(n, n-k) =& E\{(v_2[n+1] + 3v_2[n-1])(v_2[n+1-k] + 3v_2[n-1-k])^*\} \\ =& \sigma^2\delta(k) + 9\sigma^2\delta(k) + 3\sigma^2\delta(k-2) + 3\sigma^2\delta(k+2) \\ =& 5\delta(k) + 1.5\delta(k+2) + 1.5\delta(k-2) \end{aligned}$$

Thus x and y are WSS.

(b)

$$\begin{aligned} r_{xy}(n, n-k) =& E\{(v_1[n] + 3v_2[n-1])(v_2[n+1-k] + 3v_2[n-1-k])^*\} \\ =& 9\sigma^2\delta(k) + 3\sigma^2\delta(k-2) = 5\delta(k) + 1.5\delta(k-2) \\ r_{yx}(n, n-k) =& E\{(v_2[n+1] + 3v_2[n-1])(v_1[n-k] + 3v_2[n-1-k])^*\} \\ =& 9\sigma^2\delta(k) + 3\sigma^2\delta(k+2) \\ =& 5\delta(k) + 1.5\delta(k+2) \end{aligned}$$

Thus x and y are jointly WSS.

8. A random process $x(n)$ has correlation matrix

$$\mathbf{R} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) Determine the KLT basis functions.

(b) Determine the minimum mean square error achievable when using a single term approximation, e.g.,

$$\hat{x}(n) = c_1(n)\mathbf{q}_1$$

(c) Determine the error when 2, 3 and 4 terms are used.

Solution:

(a) Eigenvalues of **R**:

$$\lambda_1 = 1, \lambda_2 = \lambda_3 = 3, \lambda_4 = 5.$$

The corresponding eigenvectors:

$$
\begin{aligned}
\mathbf{q}_1 &= [0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T \\
\mathbf{q}_2 &= [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0]^T \\
\mathbf{q}_3 &= [0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T \\
\mathbf{q}_4 &= [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0]^T
\end{aligned}
$$

Hence, $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ are the KLT basis functions.

(b)

$$\hat{x}(n) = c_1(n)\mathbf{q}_1$$

$$\varepsilon(n) = x(n) - \hat{x}(n)$$

When single item is used, the MSE is given by

$$
\begin{aligned}
\varepsilon_{MSE} &= E[\varepsilon(n)^H \varepsilon(n)] \\
&= \sum_{i=1}^{4} \lambda_i - \lambda_{j, j=1,2,3,4}
\end{aligned}
$$

Choose the biggest eigenvalue, the minimum available MSE is 7.

(c) When 2 items are used, the minimum available MSE is 4. When 3 items are used, the minimum available MSE is 1. When 4 items are used, the minimum available MSE is 0.

9. Let

$$v[n] = a_1^* x[n] + a_2^* y[n]$$

where x and y are WSS processes. Show that the coherence satisfies

$$|\Gamma(\omega)|^2 = \frac{|S_{xy}(\omega)|^2}{S_x(\omega)S_y(\omega)} \geq 1$$

where $S_x(\omega), S_y(\omega), S_{xy}(\omega)$ are PSD and cross–PSD functions. HINT: Write the output PSD function using a matrix expression, $S_v(\omega) = \mathbf{a}^H \boldsymbol{S}\mathbf{a}$, and use known properties.

Solution: By definition and substitution

$$
\begin{aligned}
r_v(k) &= E\{v(n)v^*(n-k)\} \\
&= E\{(a_1^* x(n) + a_2^* y(n))(a_1 x^*(n-k) + a_2 y^*(n-k))\} \\
&= a_1 a_1^* r_x(k) + a_1^* a_2 r_{x,y}(k) + a_1 a_2^* r_{y,x}^*(-k) + a_2 a_2^* r_y(k) \\
&= a_1 a_1^* r_x(k) + a_1^* a_2 r_{x,y}(k) + a_1 a_2^* r_{y,x}(k) + a_2 a_2^* r_y(k)
\end{aligned}
$$

This can be written in matrix form

$$r_v(k) = [a_1^*, a_2^*] \begin{bmatrix} r_x(k) & r_{x,y}(k) \\ r_{y,x}(k) & r_y(k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{a}^H \mathbf{R} \mathbf{a} \qquad (*)$$

for

$$\mathbf{a}^H = [a_1^*, a_2^*] \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} r_x(k) & r_{x,y}(k) \\ r_{y,x}(k) & r_y(k) \end{bmatrix}$$

Taking the Fourier transform of $(*)$ gives,

$$S_v(\omega) = [a_1^*, a_2^*] \begin{bmatrix} S_x(\omega) & S_{x,y}(\omega) \\ S_{y,x}(\omega) & S_y(\omega) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{a}^H \mathbf{S} \mathbf{a}$$

Note that $S_v(\omega) \geq 0 \Rightarrow \mathbf{a}^H \mathbf{S} \mathbf{a} \geq 0$. And since a_1 and a_2 are arbitrary, this holds for all $\mathbf{a}$, which is true if and only if $|\mathbf{S}| \geq 0$. Thus

$$|\mathbf{S}| \geq 0 \Rightarrow S_x(\omega)S_y(\omega) - S_{x,y}(\omega)S_{y,x}(\omega) = S_x(\omega)S_y(\omega) - |S_{x,y}(\omega)|^2 \geq 0 \Rightarrow |\Gamma(\omega)| \leq 1$$

10. Let $\{x[n]\}$ be a process of i.i.d. RVs uniform on $[-1, 1]$. The process is passed through a LTI system with impulse response $h[n] = \left(\frac{1}{2}\right)^n U[n]$, yielding output $y[n]$.

    (a) Determine $R_{yx}[l]$.

    (b) Determine $R_y[l]$.

    (c) Determine $S_y(\omega)$

    Solution:

    (a) Note $R_{xx}[l] = \frac{1}{3}\delta[l] \Rightarrow R_{yx}[l] = R_{xx}[l] * h[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n U[n]$.

    (b) $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow |H(\omega)|^2 = \left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{j\omega}}\right) = \frac{1}{1.25-\cos\omega} \Rightarrow S_y(\omega) = \frac{\frac{1}{3}}{1.25-\cos\omega}$
    Inverse transform gives $R_y[l]$.

11. (a) Determine the mean of the exponential density function $f_x(x) = \alpha e^{-\alpha x}U(x)$, and expressed the density in terms of the mean parameter $\mu = E\{x\}$.

    (b) Given independence samples $x_1, x_2, \ldots, x_N$, determine the ML estimate of μ.

    (c) Is the estimate unbiased?

    (d) Is it consistent?

    (e) What is the variance of the estimate? Is it a minimum variance estimate?

    Solution:

    (a)

$$\mu = E[x] = \int_0^\infty \alpha e^{-\alpha x} dx = \frac{1}{\alpha}$$

(b)

$$\begin{aligned} f_{x_1,x_2,\cdots,x_N}(x_1,x_2,\cdots,x_N) &= \prod_{i=1}^{N} f_{x_i}(x_i) \\ &= \alpha^N e^{-\alpha\sum_{i=1}^{N} x_i} \end{aligned}$$

$$\frac{df_{x_1,x_2,\cdots,x_N}(x_1,x_2,\cdots,x_N)}{d\alpha} = N\alpha^{N-1}e^{-\sum_{i=1}^{N}\alpha} + \alpha^N\left(-\sum_{i=1}^{N} x_i\right)e^{-\sum_{i=1}^{N}\alpha} = 0$$

$$\Downarrow$$

$$\alpha = \frac{N}{\sum_{i=1}^{N} x_i}$$

So, the maximum likelihood estimate of μ is

$$\hat{\mu} = \frac{\sum_{i=1}^{N} x_i}{N}$$

(c)

$$\begin{aligned} E[\hat{\mu}] &= E\left[\frac{\sum_{i=1}^{N} x_i}{N}\right] \\ &= \frac{\sum_{i=1}^{N} E[x_i]}{N} \\ &= \frac{\sum_{i=1}^{N} \mu}{N} \\ &= \mu \end{aligned}$$

So, the estimate is unbiased.

(d)

$$\begin{aligned} E[\hat{\mu}_N^2] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)\left(\frac{1}{N}\sum_{j=1}^{N} x_j\right)\right] \\ &= \frac{1}{N^2}\left(\sum_{i=1}^{N} E[x_i^2] + \sum_{i\neq j} E[x_i]E[x_j]\right) \\ &= \frac{1}{N^2}\left(\sum_{i=1}^{N}\frac{2}{\alpha^2} + N(N-1)\frac{1}{\alpha^2}\right) \\ &= \frac{1}{\alpha^2}(1+\frac{1}{N}) \end{aligned}$$

The variance of the estimate is

$$\begin{aligned} var(\hat{\mu}_N) &= E[\hat{\mu}^2] - E[\hat{\mu}]^2 \\ &= \frac{1}{\alpha^2}(1+\frac{1}{N}) - \frac{1}{\alpha^2} \\ &= \frac{1}{N\alpha^2} \\ &= \frac{\mu^2}{N} \end{aligned}$$

$$var(\hat{\mu}_N) < var(\hat{\mu}_{N-1})$$

Also, from (3), $\hat{\mu_N}$ is unbiased.

So, $\hat{\mu_N}$ is a consistent estimate.

(e)

$$f_{\mathbf{x}|\mu}(\mathbf{x}|\mu) = \frac{1}{\mu^N} e^{-\frac{\sum_{i=1}^N x_i}{\mu}}$$

$$\ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)] = -N \ln \mu - \frac{\sum_{i=1}^N x_i}{\mu}$$

$$\frac{\partial^2}{\partial \mu^2} \ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)] = \frac{N}{\mu^2} - \frac{2\sum_{i=1}^N x_i}{\mu^3}$$

$$E\left[\frac{\partial^2}{\partial \mu^2} \ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)]\right] = -\frac{N}{\mu^2}$$

$$\left(-E\left[\frac{\partial^2}{\partial \mu^2} \ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)]\right]\right)^{-1} = \frac{\mu^2}{N} = var(\hat{\mu}_N)$$

That is, the estimate $\hat{\mu}_N = \frac{\sum_{i=1}^N x_i}{N}$ hits the Cramer-Rao Bound.

So, the unbiased estimate $\hat{\mu}_N$ is a minimum variance estimate.

12. The joint density function of random variables x and y is given by

$$f_{xy}(x,y) \begin{cases} 6x & 0 \le x \le 1; 0 \le y \le 1-x \\ 0 & \text{otherwise} \end{cases}$$

(a) Determined and sketch the conditional density function $f_{y|x}(y|x)$.

(b) Determine the MAP estimate of y.

(c) Determine the MS estimate of y.

(d) Determine the MAE estimate of y.

13. A two-dimensional vector $\mathbf{x}$ and a random variable y have the joint density

$$f_{\mathbf{x}y}(\mathbf{x}, y) \begin{cases} (y + 3x_1)x_2 & 0 \leq x_1, x_2, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $\mathbf{x}$ represents the observation.

(a) Determine the MAP estimate of y.
(b) Determine the MS estimate of y.
(c) Determine the MAE estimate of y.

Solution:

(a)

$$f_{\mathbf{x}}(\mathbf{x}) = \int f_{\mathbf{x},y}(\mathbf{x}, y)dy = \left(\frac{1}{2} + 3x_1\right)x_2, \quad 0 \leq x_1,\ x_2 \leq 1$$
$$\Rightarrow f_{y|\mathbf{x}}(y|\mathbf{x}) = \frac{f_{\mathbf{x},y}(\mathbf{x}, y)}{f_{\mathbf{x}}(\mathbf{x})} = \frac{y + 3x_1}{\frac{1}{2} + 3x_1}, \quad 0 \leq x_1,\ y \leq 1$$
$$\Rightarrow \hat{y}_{\text{MAP}} = \arg\max_y f_{y|\mathbf{x}}(y|\mathbf{x}) = 1$$

(b)

$$\hat{y}_{\text{MS}} = E\{y|\mathbf{x}\} = \int_0^1 y \frac{y + 3x_1}{\frac{1}{2} + 3x_1} dy = \frac{2 + 9x_1}{3 + 18x_1}$$

(c)

$$F_{y|\mathbf{x}}(y|\mathbf{x}) = \int_0^{\hat{y}_{MAE}} \frac{y + 3x_1}{\frac{1}{2} + 3x_1} dy = \frac{1}{2} \quad \Rightarrow \quad \frac{\hat{y}_{MAE}^2 + \hat{y}_{MAE} 6x_1}{1 + 6x_1} = \frac{1}{2}$$

Therefore, $\hat{y}_{\text{MAE}} = \frac{-6x_1 \pm \sqrt{36x_1^2 + 12x_1 + 2}}{2}$.

14. A random process $x[n]$ is generated according to the difference equation

$$x[n] = \rho x[n-1] + \eta[n]$$

where ρ is a constant and is a binary whitenoise sequence taking on values -1 and $+1$ with equal probabilities.

(a) Generate and plot $M = 50$ samples of the random sequence for $\rho = 0.95, 0.70$, and -0.95. What differences do you observe in these three random sequences?
(b) Repeat the above with $\eta[n]$ a white noise Gaussian sequence with unit variance.
(c) Let $\hat{R}_x[l]$ be the sample autocorrelation. Define the estimated correlation coefficient $\hat{\rho}$ as

$$\hat{\rho} = \frac{\hat{R}_x[1]}{\hat{R}_x[0]}$$

Compute $\hat{\rho}$ for each of the Gaussian noise driven sequences and for several sequence lengths. How well does the estimated value compare with the theoretical value?
(d) Plot the estimated $\hat{R}_x[l]$ and true $R_x[l]$ autocorrelation functions for $0 \leq l \leq 1$ for each of the Gaussian noise driven sequences.
(e) What happens if $|\rho| > 1$?