1. A random process $x(n)$ has correlation matrix

$$\mathbf{R} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   (a) Determine the KLT basis functions.

   (b) Determine the minimum mean square error achievable when using a single term approximation, e.g.,

$$\hat{x}(n) = c_1(n)\mathbf{q}_1$$

   (c) Determine the error when 2, 3 and 4 terms are used.

Solution.
(a) Eigenvalues of **R**:

$$\lambda_1 = 1, \lambda_2 = \lambda_3 = 3, \lambda_4 = 5.$$

The corresponding eigenvectors:

$$\begin{aligned} \mathbf{q}_1 &= [0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T \\ \mathbf{q}_2 &= [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0]^T \\ \mathbf{q}_3 &= [0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T \\ \mathbf{q}_4 &= [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0]^T \end{aligned}$$

Hence, $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ are the KLT basis functions.

(b)

$$\hat{x}(n) = c_1(n)\mathbf{q}_1$$
$$\varepsilon(n) = x(n) - \hat{x}(n)$$

When single item is used, the MSE is given by

$$\begin{aligned} \varepsilon_{MSE} &= E[\varepsilon(n)^H \varepsilon(n)] \\ &= \sum_{i=1}^{4} \lambda_i - \lambda_{j, j=1,2,3,4} \end{aligned}$$

Choose the biggest eigenvalue, the minimum available MSE is 7.

(c) When 2 items are used, the minimum available MSE is 4. When 3 items are used, the minimum available MSE is 1. When 4 items are used, the minimum available MSE is 0.

2. A real random process is defined by

$$x(n) = A\cos\omega_o n + w(n)$$

where A is a Gaussian random variable with mean zero and variance σ_A^2 and $w(n)$ is a white noise process independent of A with variance σ_w^2.

   (a) What is the correlation function of $x(n)$?

   (b) Can the power spectral density of $x(n)$ be defined? If so, what is the power spectral density function?

   (c) Repeat (a) and (b) in the case when the cosine has an independent random phase uniformly distributed on $[-\pi, \pi]$.

Solution.
(a) The correlation function $r_x(n, n-k)$

$$\begin{aligned} r_x(n, n-k) &= E\{x(n)x^*(n-k)\} \\ &= \sigma_A^2 \cos(\omega_0 n)\cos(\omega_0(n-k)) + \sigma_\omega^2\delta(k) \\ &= \frac{\sigma_A^2}{2}[\cos(\omega_0 k) + \cos(\omega_0(2n-k))] + \sigma_\omega^2\delta(k) \end{aligned}$$

(b)Since the correlation function $r_x(n, n-k)$ is not independent with n, the process is not widesense stationary. Therefore, the power spectral density function can not be defined.

(c)If

$$x(n) = A\cos(\omega_0 n + \theta)\ , \quad \theta \in \{-\pi, \pi\}$$

Then,

$$\begin{aligned} r_x(n, n-k) &= E\{x(n)x^*(n-k)\} \\ &= \frac{\sigma_A^2}{2}\cos(\omega_0 k) + \sigma_\omega^2\delta(k) + \frac{\sigma_A^2}{2}\underbrace{E[\cos(\omega_0(2n-k) + 2\theta)]}_{0} \\ &= \frac{\sigma_A^2}{2}\cos(\omega_0 k) + \sigma_\omega^2\delta(k) \end{aligned}$$

Since the correlation function $r_x(n, n-k)$ only depends on k, this process is widesense stationary. Therefore, we can define the power spectral density function of $x(n)$.

$$S_x(\omega) = \frac{\sigma_A^2}{4}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \sigma_\omega^2$$

3. A linear system is defined by

$$y(n) = 0.7y(n-1) + x(n) - x(n-1)$$

   (a) Compute the first four values of $R_{yx}(l)$ it is known that $R_x(l) = \delta(l)$.

(b) What is $R_{yx}(l)$ for $-3 \leq l \leq 3$?

(c) What is the power spectral density function $S_y(\omega)$?

Solution.

(a)

$$y(n) = 0.7y(n-1) + x(n) - x(n-1)$$
$$Y(z) = 0.7z^{-1}Y(z) + X(z) - z^{-1}X(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-0.7^{-1}}$$
$$h(n) = 0.7^n u(n) - 0.7^{n-1}u(n-1)$$

Since

$$\begin{aligned} R_{yx}(l) &= h(l) * R_x(l) \\ &= h(l) * \delta(l) \\ &= h(l) \end{aligned}$$

Therefore, the first four values of $R_{yx}(l)$ are $\{1, -0.3, -0.21, -0.147\}$.

(b)$R_{yx}(l) = \{0, 0, 0, 1, -0.3, -0.21, -0.147\}$, $-3 \leq l \leq 3$.

(c)

$$\begin{aligned} R_y(l) &= h(l) * h^*(-l) * R_x(l) \\ &= h(l) * h^*(-l) \end{aligned}$$

Then,

$$\begin{aligned} S_y(\omega) &= |H(\omega)|^2 \\ &= \left| \frac{1-e^{-j\omega}}{1-0.7e^{-j\omega}} \right|^2 \\ &= \frac{2-2\cos(\omega)}{1.49-1.4\cos(\omega)} \end{aligned}$$

4. A random process is defined by

$$x(n) = s(n) + \eta(n)$$

where $\eta(n)$ is a unit variance white noise process and $s(n)$ is defined by

$$s(n) = \rho s(n-1) + w(n)$$

where $w(n)$ is another unit variance white noise process independent of $\eta(n)$.

(a) Determine the correlation function $R_x(l)$.

(b) Determine the power spectral density function $S_x(\omega)$.

Solution.
(a)

$$\begin{aligned} R_x(l) &= E\{x(n)x^*(n-l)\} \\ &= E\{[s(n)+\eta(n)][s(n-l)+\eta(n-l)]^*\} \\ &= R_s(l)+R_\eta(l) \end{aligned}$$

and

$$\begin{aligned} R_s(l) &= E\{s(n)s^*(n-l)\} \\ &= E\{[\rho s(n-1)+w(n)]s^*(n-l)\} \\ &= \rho R_s(l-1) \end{aligned}$$

Since

$$\begin{aligned} R_s(0) &= E\{[\rho s(n-1)+w(n)][\rho s(n-1)+w(n)]^{ast}\} \\ &= \rho^2 R_s(0)+1 \end{aligned}$$

,

$$R_s(0) = \frac{1}{1-\rho^2}$$

Therefore

$$R_s(l) = \frac{\rho^{|l|}}{1-\rho^2}$$

Then

$$R_x(l) = \frac{\rho^{|l|}}{1-\rho^2}+\delta(l)$$

(b)

$$S_x(\omega) = 1+\frac{1}{1+\rho^2-2\rho\cos(\omega)}$$

5. Recall the system

$$y(n) = \mathbf{w}^T\mathbf{x}(n)$$

where $\mathbf{x}(n) = \mathbf{u}(n)+\mathbf{v}(n)$. It was proven in class that the matched filter for a deterministic signal, corrupted by white noise, is given

$$\mathbf{w} = k\mathbf{u}^*(n)$$

where $\mathbf{w}$ is the vector of filter coefficients and $\mathbf{u}(n)$ is the deterministic signal. Suppose now the noise is colored (i.e.$\mathbf{R}_v \neq \sigma_v^2\mathbf{I}$). Utilize the transformation

$$\mathbf{x}'(n) = k\mathbf{R}_v^{-1/2}\mathbf{x}(n)$$

to determine a similar result for the colored noise case.

Solution.

$$\begin{aligned}
\text{SNR} &= \frac{|y_s(n)|^2}{E\{|y_n(n)|^2\}} \\
&= \frac{\mathbf{w}^H\mathbf{u}^*(n)\mathbf{u}^T(n)\mathbf{w}}{E\{\mathbf{w}^H\mathbf{v}^*(n)\mathbf{v}^T(n)\mathbf{w}\}} \\
&= \frac{|\mathbf{w}^H\mathbf{u}^*(n)|^2}{\mathbf{w}^H\mathbf{R}_v\mathbf{w}}
\end{aligned}$$

Since $\mathbf{R}_v$ is Hermitian and positive definite, $\mathbf{R}_v = \mathbf{L}\cdot\mathbf{L}^H$.

$$\begin{aligned}
\text{SNR} &= \frac{|\mathbf{w}^H\mathbf{L}\mathbf{L}^{-1}\mathbf{u}^*(n)|^2}{\mathbf{w}^H\mathbf{L}\mathbf{L}^H\mathbf{w}} \\
&= \underbrace{\frac{|(\mathbf{L}^H\mathbf{w})^H\cdot(\mathbf{L}^{-1}\mathbf{u}^*(n))|^2}{(\mathbf{L}^H\mathbf{w})^H\mathbf{L}^H\mathbf{w}}}_{\text{Cauchy-Schwarz inequality}} \\
&\leq \frac{|\mathbf{L}^H\mathbf{w}|^2\cdot|\mathbf{L}^{-1}\mathbf{u}^*(n)|^2}{|\mathbf{L}^H\mathbf{w}|^2} \\
&= |\mathbf{L}^{-1}\mathbf{u}^*(n)|^2
\end{aligned}$$

The maximum SNR is achieved when

$$\begin{aligned}
\mathbf{L}^H\mathbf{w} &= k\mathbf{L}^{-1}\mathbf{u}^*(n) \\
\mathbf{w} &= k\mathbf{R}_v^{-1}\mathbf{u}^*(n)
\end{aligned}$$

6. (a) Determine the mean of the exponential density function $f_x(x) = \alpha e^{-\alpha x}U(x)$, and expressed the density in terms of the mean parameter $\mu = E\{x\}$.
   (b) Given independence samples $x_1, x_2, \ldots, x_N$, determine the ML estimate of μ.
   (c) Is the estimate unbiased?
   (d) Is it consistent?
   (e) What is the variance of the estimate? Is it a minimum variance estimate?

Solution.

(a)

$$\mu = E[x] = \int_0^\infty \alpha e^{-\alpha x}dx = \frac{1}{\alpha}$$

(b)

$$\begin{aligned}
f_{x_1,x_2,\cdots,x_N}(x_1,x_2,\cdots,x_N) &= \prod_{i=1}^N f_{x_i}(x_i) \\
&= \alpha^N e^{-\alpha\sum_{i=1}^N x_i}
\end{aligned}$$

$$\frac{df_{x_1,x_2,\cdots,x_N}(x_1,x_2,\cdots,x_N)}{d\alpha} = N\alpha^{N-1}e^{-\sum_{i=1}^{N}\alpha} + \alpha^N\left(-\sum_{i=1}^{N}x_i\right)e^{-\sum_{i=1}^{N}\alpha} = 0$$

$$\Downarrow$$

$$\alpha = \frac{N}{\sum_{i=1}^{N}x_i}$$

So, the maximum likelihood estimate of μ is

$$\hat{\mu} = \frac{\sum_{i=1}^{N}x_i}{N}$$

(c)

$$\begin{aligned} E[\hat{\mu}] &= E\left[\frac{\sum_{i=1}^{N}x_i}{N}\right] \\ &= \frac{\sum_{i=1}^{N}E[x_i]}{N} \\ &= \frac{\sum_{i=1}^{N}\mu}{N} \\ &= \mu \end{aligned}$$

So, the estimate is unbiased.

(d)

$$\begin{aligned} E[\hat{\mu}_N^2] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)\left(\frac{1}{N}\sum_{j=1}^{N}x_j\right)\right] \\ &= \frac{1}{N^2}\left(\sum_{i=1}^{N}E[x_i^2] + \sum_{i\neq j}E[x_i]E[x_j]\right) \\ &= \frac{1}{N^2}\left(\sum_{i=1}^{N}\frac{2}{\alpha^2} + N(N-1)\frac{1}{\alpha^2}\right) \\ &= \frac{1}{\alpha^2}(1+\frac{1}{N}) \end{aligned}$$

The variance of the estimate is

$$\begin{aligned} var(\hat{\mu}_N) &= E[\hat{\mu}^2] - E[\hat{\mu}]^2 \\ &= \frac{1}{\alpha^2}(1+\frac{1}{N}) - \frac{1}{\alpha^2} \\ &= \frac{1}{N\alpha^2} \\ &= \frac{\mu^2}{N} \end{aligned}$$

$$var(\hat{\mu}_N) < var(\hat{\mu}_{N-1})$$

Also, from (3), $\hat{\mu_N}$ is unbiased.

So, $\hat{\mu_N}$ is a consistent estimate.

(e)

$$f_{\mathbf{x}|\mu}(\mathbf{x}|\mu) = \frac{1}{\mu^N} e^{-\frac{\sum_{i=1}^N x_i}{\mu}}$$

$$\ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)] = -N \ln \mu - \frac{\sum_{i=1}^N x_i}{\mu}$$

$$\frac{\partial^2}{\partial \mu^2} \ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)] = \frac{N}{\mu^2} - \frac{2\sum_{i=1}^N x_i}{\mu^3}$$

$$E\left[\frac{\partial^2}{\partial \mu^2} \ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)]\right] = -\frac{N}{\mu^2}$$

$$\left(-E\left[\frac{\partial^2}{\partial \mu^2} \ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)]\right]\right)^{-1} = \frac{\mu^2}{N} = var(\hat{\mu}_N)$$

That is, the estimate $\hat{\mu}_N = \frac{\sum_{i=1}^N x_i}{N}$ hits the Cramer-Rao Bound.

So, the unbiased estimate $\hat{\mu}_N$ is a minimum variance estimate.

7. The joint density function of random variables x and y is given by

$$f_{xy}(x,y) \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determined and sketch the conditional density function $f_{y|x}(y|x)$.
   (b) Determine the MAP estimate of y.
   (c) Determine the MS estimate of y.
   (d) Determine the MAE estimate of y.

Solution.

(a)

$$f(x) = \int_0^x 8xydy = 4x^3 \quad 0 \le x \le 1$$

$$f_{y|x}(y|x) = \frac{f_{x,y}(x,y)}{f(x)} = \frac{2y}{x^2} \quad 0 \le y \le x$$

(b) $\hat{y}_{\text{MAP}} = x$

(c)

$$\begin{aligned} \hat{y}_{\text{MS}} &= E\{y|x\} \\ &= \int_0^x y\frac{2y}{x^2}dy \\ &= \frac{2x}{3} \end{aligned}$$

(d)

$$F_{y|x}(y|x) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{x^2} & 0 \le y \le x \\ 1 & y > x \end{cases}$$

Therefore, $\hat{y}_{\text{MAE}} = \frac{x}{\sqrt{2}}$.

8. A two-dimensional vector $\mathbf{x}$ and a random variable y have the joint density

$$f_{\mathbf{x}y}(\mathbf{x}, y) \begin{cases} A(y + 6x_1)x_2 & 0 \le x_1, x_2, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $\mathbf{x}$ represents the observation.

(a) Determine the MAP estimate of y.
(b) Determine the MS estimate of y.
(c) Determine the MAE estimate of y.

Solution.
(a)

$$f_{\mathbf{X}}(\mathbf{x}) = \int f_{\mathbf{X},y}(\mathbf{x}, y)dy = A(\frac{1}{2} + 6x_1)x_2, \quad 0 \le x_1,\ x_2 \le 1$$

$$f_{y|\mathbf{X}}(y|\mathbf{x}) = \frac{f_{\mathbf{X},y}(\mathbf{x}, y)}{f_{\mathbf{X}}(\mathbf{x})} = \frac{y + 6x_1}{1/2 + 6x_2}, \quad 0 \le x_1,\ y \le 1$$

$$\hat{y}_{\text{MAP}} = \arg\max_y f_{y|\mathbf{X}}(y|\mathbf{x}) = 1$$

(b)

$$\hat{y}_{\text{MS}} = E\{y|\mathbf{x}\} = \int_0^1 y \frac{y + 6x_1}{1/2 + 6x_2} dy = \frac{2 + 18x_1}{3 + 36x_2}$$

(c)

$$F_{y|\mathbf{x}}(y|\mathbf{x}) = \int_0^{\hat{y}_{MAE}} \frac{y + 6x_1}{1/2 + 6x_2} dy = \frac{1}{2}$$

Therefore, $\hat{y}_{\text{MAE}} = -6x_1 + \sqrt{36x_1^2 + 6x_2 + 1/2}$.

9. A random process $x[n]$ is generated according to the difference equation

$$x[n] = \rho x[n-1] + \eta[n]$$

where ρ is a constant and is a binary whitenoise sequence taking on values -1 and $+1$ with equal probabilities.

(a) Generate and plot $M = 50$ samples of the random sequence for $\rho = 0.95, 0.70$, and -0.95. What differences do you observe in these three random sequences?

(b) Repeat the above with $\eta[n]$ a white noise Gaussian sequence with unit variance.

(c) Let $\hat{R}_x[l]$ be the sample autocorrelation. Define the estimated correlation coefficient $\hat{\rho}$ as

$$\hat{\rho} = \frac{\hat{R}_x[1]}{\hat{R}_x[0]}$$

Compute $\hat{\rho}$ for each of the Gaussian noise driven sequences and for several sequence lengths. How well does the estimated value compare with the theoretical value?

(d) Plot the estimated $\hat{R}_x[l]$ and true $R_x[l]$ autocorrelation functions for $0 \leq l \leq 1$ for each of the Gaussian noise driven sequences.

(e) What happens if $|\rho| > 1$?