1. A signal with correlation function $R_x[l] = \rho^{|l|}$ is applied to a linear shift-invariant system with impulse response $h[n] = \delta[n] - \delta[n-1]$.

- Compute and plot the cross–correlation function $R_{yx}[l]$.
- Compute and plot the output correlation function $R_y[l]$.

Answer:
Recall that

$$f[n] * \delta[n - n_0] = f[n - n_0]$$

So,

$$\begin{aligned} R_{yx}[l] &= h[l] * R_x[l] \\ &= (\delta[n] - \delta[n-1]) * R_x[l] \\ &= \delta[n] * R_x[l] - \delta[n-1] * R_x[l] \\ &= R_x[l] - R_x[l-1] \\ &= \rho^{|l|} - \rho^{|l-1|} \end{aligned}$$

$$\begin{aligned} R_y[l] &= h[l] * h^[- l] * R_x[l] \\ &= h[l] * (\delta[-l] - \delta[-l-1]) * R_x[l] \\ &= h[l] * (R_x[-l] - R_x[-l-1]) \\ &= (\delta[n] - \delta[n-1]) * (R_x[-l] - R_x[-l-1]) \\ &= -R_x[-l-1] + 2R_x[-l] - R_x[-l+1] \\ &= -\rho^{|-l-1|} + 2\rho^{|-l|} - \rho^{|-l+1|} \end{aligned}$$

2. In a matched filter application the the SNR is given by

$$SNR = \frac{\mathbf{w}^H \mathbf{s}^* \mathbf{s}^T \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}}$$

which reduces to

$$SNR = \frac{\mathbf{w}^H \mathbf{s}^* \mathbf{s}^T \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}}$$

when $\mathbf{R}_n = \sigma^2 \mathbf{I}$, yielding the solution $\mathbf{w} = k\mathbf{s}^*$. Show that for colored noise ($\mathbf{R}_n \neq \sigma^2 \mathbf{I}$) that using the transformation

$$\mathbf{x}' = \mathbf{R}_n^{-1/2} \mathbf{x}$$

reduces the problem to the white noise case with solution $\mathbf{w} = k\mathbf{R}^{-1}\mathbf{s}^*$.

Answer:
Let the input to the system be

$$\mathbf{x} = \mathbf{s} + \mathbf{n}$$

where $\mathbf{x}$ is the signal and $\mathbf{x}$ is the noise.

$$\mathbf{x}' = \mathbf{R}_n^{-1/2} \mathbf{x}$$

$$\Downarrow$$
$$\mathbf{x} = \mathbf{R}_n^{1/2}\mathbf{x}'$$

Let

$$\mathbf{s} = \mathbf{R}_n^{1/2}\mathbf{s}'$$

and

$$\mathbf{n} = \mathbf{R}_n^{1/2}\mathbf{n}'$$

So

$$\begin{aligned} SNR &= \frac{\mathbf{w}^H(\mathbf{R}_n^{1/2}\mathbf{s}')^*(\mathbf{R}_n^{1/2}\mathbf{s}')^T\mathbf{w}}{\mathbf{w}^H\mathbf{R}\mathbf{w}} \\ &= \frac{\left((\mathbf{R}_n^{1/2})^T\mathbf{w}\right)^H\mathbf{s}'^*\mathbf{s}'^T(\mathbf{s}')^T\left((\mathbf{R}_n^{1/2})^T\mathbf{w}\right)}{\mathbf{w}^H\mathbf{R}\mathbf{w}} \end{aligned}$$

Since $\mathbf{R}$ is positive definite, according to Cholesky Decomposition method, there is a matrix $\mathbf{A}$ such that

$$\mathbf{R} = \mathbf{A}\mathbf{A}^H$$

Up to now, we have not defined $\mathbf{R}_n^{1/2}$.

Now we define $\mathbf{R}_n^{1/2} = A^*$, then

$$\mathbf{R} = (\mathbf{R_n^{1/2}})^*(\mathbf{R_n^{1/2}})^{*H} = (\mathbf{R_n^{1/2}})^*(\mathbf{R_n^{1/2}})^T$$

Thus,

$$\begin{aligned} SNR &= \frac{\left((\mathbf{R}_n^{1/2})^T\mathbf{w}\right)^H\mathbf{s}'^*\mathbf{s}'^T(\mathbf{s}')^T\left((\mathbf{R}_n^{1/2})^T\mathbf{w}\right)}{\mathbf{w}^H(\mathbf{R_n^{1/2}})^*(\mathbf{R_n^{1/2}})^T\mathbf{w}} \\ &= \frac{\left((\mathbf{R}_n^{1/2})^T\mathbf{w}\right)^H\mathbf{s}'^*\mathbf{s}'^T(\mathbf{s}')^T\left((\mathbf{R}_n^{1/2})^T\mathbf{w}\right)}{\left((\mathbf{R_n^{1/2}})^\mathbf{T}\mathbf{w}\right)^H\left((\mathbf{R_n^{1/2}})^T\mathbf{w}\right)} \end{aligned}$$

So, the SNR is maximized, when

$$(\mathbf{R}_n^{1/2})^T\mathbf{w} = k(\mathbf{s}')^*$$

or when

$$(\mathbf{R}_n^{1/2})^T\mathbf{w} = k(\mathbf{R}_n^{-1/2})^*\mathbf{s}^*$$

$$\begin{aligned} \mathbf{w} &= k(\mathbf{R}_n^{-1/2})^T(\mathbf{R}_n^{-1/2})^*\mathbf{s}^* \\ &= k\mathbf{R}^{-1}\mathbf{s}^* \end{aligned}$$

3. A random variable x has the uniform density

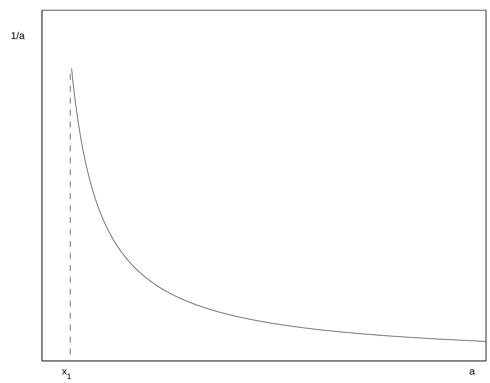
$$f_x(x) = \begin{cases} 1/a & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

- Determine the likelihood function $f_{x_1,x_2,\cdots,x_N;a}$ for $N=1$ and $N=2$ and sketch it. Find the maximum likelihood estimate of the parameter a for these two cases.
- Determine the maximum likelihood estimate of the parameter a for arbitrary N.
- Show that the maximum likelihood estimate for the parameter a is a biased estimate for $N=1$ and $N=2$.
- Find the expected value of the maximum likelihood estimate of a for arbitrary N and show that the estimate is asymptotically unbiased.

Answer:
(1) For $N=1$

$$\begin{aligned} f_{x_1;a} &= f_x(x_1) \\ &= \begin{cases} 1/a & 0 \le x_1 \le a \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

To maximize the likelihood function, $a = x_1$.



(2) For $N=2$

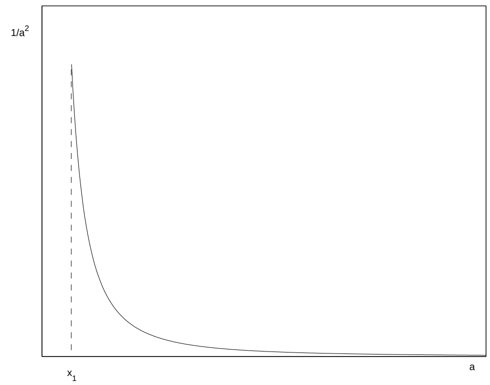
$$\begin{aligned} f_{x_1,x_2;a} &= f_x(x_1)f_x(x_2) \\ &= \begin{cases} 1/a^2 & 0 \le x_1 \le a \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

To maximize the likelihood function, $a = \max(x_1, x_2)$.
(3) The expectation of the RV a for $N=1$ is,

$$E[a_1] = E[x_1] = a/2$$

$$\text{bias} = E[a_1] - a = -a/2 \neq 0$$

Thus it is a biased estimate.



For $N = 2$, let $a_2 = \max(x_1, x_2)$,

$$\begin{aligned} F_{a_2}(x) &= P[\max(x_1, x_2) \le x] \\ &= P[x_1 \le x, x_2 \le x] \\ &= F(x)F(x) \end{aligned}$$

$$f_{a_2}(x) = \frac{dF_{a_2}(x)}{dx} = 2F(x)f(x)$$

Since

$$f_x(x) = \begin{cases} 1/a & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

So

$$F_x(x) = \begin{cases} 0 & x < 0 \\ x/a & 0 \le x \le a \\ 1 & x > a \end{cases}$$

Thus,

$$f_{a_2}(x) = 2F(x)f(x) = \begin{cases} 2x/a^2 & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$E[a_2] = \int_0^a \frac{2x}{a^2} dx = 2a/3$$

$$\text{bias} = E[a_2] - a = -a/3 \neq 0$$

Thus, the estimate for $N = 2$ is better than that for $N = 1$.

(4) For arbitrary N, let $a_N = \max(x_1, x_2, \cdots, x_N)$,

$$\begin{aligned} F_{a_N}(x) &= P[\max(x_1, x_2, \cdots, x_N) \le x] \\ &= P[x_1 \le x, x_2 \le x, \cdots, x_N \le x] \\ &= F_x(x)^N \end{aligned}$$

$$f_{a_N}(x) = \frac{dF_{a_N}(x)}{dx} = N[F_x(x)]^{N-1}f(x)$$

Using $F_x(x)$ obtained before, we have

$$f_{a_N}(x) = N[F_x(x)]^{N-1}f(x) = \begin{cases} \frac{Nx^{N-1}}{a^N} & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$E[a_N] = \int_0^a \frac{Nx^{N-1}}{a^N}dx = \frac{N}{N+1}a$$

$$\text{bias} = E[a_N] - a = -\frac{a}{N+1} \neq 0$$

$$\lim_{N\to\infty} \text{bias} = 0$$

Thus, the estimate for arbitrary N is asymptotically unbiased.

4. A random variable x is given by $x = y + \eta$ where y is uniformly distributed over $[-\alpha, \alpha]$ and η is uniformly distributed over $[-\beta, \beta]$. Assume $|\alpha| < |\beta|$.

- Compute and sketch the density functions $f_{x|y}$, f_{xy}, f_x, and $f_{y|x}$.
- The random variable x is observed. If η is considered noise, what is the mean square estimate of y?
- What is the MAP estimate of y? Does a unique estimate exist?

Answer: (1)

$$f_{x|y} = \begin{cases} \frac{1}{2\beta} & y-\beta \le x \le y+\beta \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_{xy}(x,y) &= f_{x|y}f_y(y) \\ &= \begin{cases} \frac{1}{4\alpha\beta} & y-\beta \le x \le y+\beta, \quad -\alpha \le y \le \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Since y and η are assumed to be independent, so

$$\begin{aligned} f_x(x) &= f_y(x) * f_\eta(x) \\ &= \int_{-\infty}^{\infty} f_y(x-z)f_\eta(z)dz \\ &= \int_{x-\alpha}^{x+\alpha} \frac{1}{2\alpha} f_\eta(z)dz \\ &= \begin{cases} \frac{1}{4\alpha\beta}(x+\alpha+\beta) & -\alpha-\beta \le x \le \alpha-\beta \\ \frac{1}{2\beta} & \alpha-\beta \le x \le -\alpha+\beta \\ \frac{1}{4\alpha\beta}(-x-\alpha+\beta) & -\alpha+\beta \le x \le \alpha+\beta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$f_{y|x} = \frac{f_{xy}(x,y)}{f_x(x)}$$

5. A two-dimensional random vector $\mathbf{x}$ and a random variable y have the joint density function

$$f_{\mathbf{x},y}(\mathbf{X}, Y) = \begin{cases} (Y + 3X_1)X_2 & 0 \le X_1, X_2, Y \le 1 \\ 0 & \text{else} \end{cases}$$

Assume that $\mathbf{x}$ represents the observation.

- Find the mean–square estimate of y.
- Find the MAP estimate of y.

Answer:

$$\begin{aligned} f_{\mathbf{x}}(\mathbf{x}) &= \int_{-\infty}^{\infty} f_{\mathbf{x},y}(\mathbf{x}, y)dy \\ &= \int_0^1 (y + 3x_1)x_2 dy \\ &= x_2/2 + 3x_1x_2 \end{aligned}$$

$$\begin{aligned} f_{y|\mathbf{x}}(y|\mathbf{x}) &= \frac{f_{\mathbf{x},y}(\mathbf{x}, y)}{f_{\mathbf{x}}(\mathbf{x})} \\ &= \frac{y + 3x_1}{1/2 + 3x_1} \end{aligned}$$

So, the MSE estimate is

$$\begin{aligned} E[y|\mathbf{x}] &= \int_0^1 y f_{y|\mathbf{x}}(y|\mathbf{x})dy \\ &= \int_0^1 y \frac{y + 3x_1}{1/2 + 3x_1} dy \\ &= \frac{2 + 9x_1}{3(1 + 6x_1)} \end{aligned}$$

$$\hat{y}_{\text{MAP}} = \underset{y}{\operatorname{argmax}}\, f_{y|\mathbf{x}}(y|\mathbf{x}) = \underset{y}{\operatorname{argmax}}\, \frac{y + 3x_1}{1/2 + 3x_1} = 1$$

6. A random process $x[n]$ is generated according to the difference equation

$$x[n] = \rho x[n-1] + \eta[n]$$

where ρ is a constant and $\eta[n]$ is a binary white noise sequence taking on values -1 and $+1$ with equal probabilities.

- Generate and plot $M = 50$ samples of the random sequence for $\rho = 0.95, 0.70$, and -0.95. What differences do you observe in these three random sequences?
- Repeat the above with $\eta[n]$ a white noise Gaussian sequence with unit variance.
- Let $\hat{R}_x[l]$ be the sample autocorrelation. Define the estimated correlation coefficient $\hat{\rho}$ as

$$\hat{\rho} = \frac{\hat{R}_x[1]}{\hat{R}_x[0]}$$

  Compute $\hat{\rho}$ for each of the Gaussian noise driven sequences and for several sequence lengths. How well does the estimated value compare with the theoretical value?
- Plot the estimated $\hat{R}_x[l]$ and true $R_x[l]$ autocorrelation functions for $0 \leq l \leq 1$ for each of the Gaussian noise driven sequences.
- What happens if $|\rho| > 1$?

7. Generate the KLT transform of an image. (You may select any 512×512 image or use one from the /usa/images/ directory.) Plot the reconstruction MSE vs. the number of basis functions used in the 5×5 and 7×7 observation window cases. For the 5×5 case, also show each of the image approximations.

8. We want to generate samples of a Gaussian process with autocorrelation $r_x(l) = (\frac{1}{2})^{|l|} + (-\frac{1}{2})^{|l|}$ for all l. Find the difference equation that generates the process x(n) when excited by w(n)∼ WGN(0,1)

Answer:

From the autocorrelation, we can see that $x(n)$ has no dependence with $x(n-1)$ but does have a dependence on $x(n-2)$. We can assume $x(n)$ to have the following form

$$x(n) = ax(n-2) + bw(n)$$

Solving for a and b

$$\begin{aligned} r_x(0) &= a^2 E\{x(n-2)^2\} + b^2 E\{w(n)^2\} + 2abE\{x(n-2)w(n)\} \\ &= a^2 r_x(0) + b^2 \end{aligned}$$

$$\begin{aligned} r_x(2) &= aE\{x(n-2)^2\} + bE\{w(n)x(n-2)w(n)\} \\ &= ar_x(0) \end{aligned}$$

which results in $a = \frac{1}{4}$ and $b = \frac{15}{8}$.

Checking to see that $x(n)$ has no dependence on $x(n-1)$

$$r_x(1) = E\{x(n)x(n-1) = ar_x(1) = 0\}$$

So,

$$x(n) = \frac{1}{4}x(n-2) + \frac{15}{8}w(n)$$

9. Given the AR process x(n)=x(n-1)-0.5x(n-2)+w(n), complete the following tasks.

- Determine $\rho_x(1)$
- Using $\rho_x(0)$ and $\rho_x(1)$, compute $\{\rho_x(l)\}_2^{15}$ by the corresponding difference equation
- Plot $\rho_x(l)$ and use the resulting graph to estimate its period
- Compared the period obtained in part(c) with the value obtained using the PSD of the model (Hinit: Use the frequency of the PSD peak.)

10. Consider an AR(2) process x(n) with d_0=1, a_1=-1.6454, a_2=0.9025, and w(n)$\sim$WGN(0,1).

- Generate 100 samples of the process and use them to estimate the ACS$\hat{\rho_x}(l)$, using the following equation

$$\hat{\rho}(l) = \frac{\sum_{n=l}^{N-1} x(n)x^*(n-l)}{\sum_{n=0}^{N-1} |x(n)|^2}$$

- Plot and compare the estimated and theoretical ACS values for 0$\leq l \leq$10
- Use the estimated value of $\rho_x(l)$ and the Yule-Walker equations to estimate the parameters of the model. Compare the estimated with the true values, and component on the accuracy of the approach.
- Use the estimated parameters to compute the PSDs of the process
- Compute and compare the estimated with the true PACS.

11. Show that if R is the correlation matrix of the random vector $\mathbf{X} : [\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}]$ and R^{-1} is its inverse, then $E\left\{\mathbf{X}R^{-1}\mathbf{X}^t\right\} = n$

Answer:
Note that $\mathbf{X}$ is a row vector. To avoid confusion, we define

$$\mathbf{Y} = \mathbf{X}^t$$

So, we need to prove that

$$E\left\{\mathbf{Y}^t R^{-1} \mathbf{Y}\right\} = n$$

Recall that

$$\mathbf{R} = \mathbf{Q}\mathbf{\Omega}^{-1}\mathbf{Q}^H = \sum_{i=1}^{n} \frac{1}{\lambda_i}\mathbf{q}_i\mathbf{q}_i^t$$

where λ_i and $\mathbf{q}_i$ are the eigenvalues and the orthonormal eigenvectors of $\mathbf{R}$.

$$\begin{aligned} E\left\{\mathbf{Y}^t R^{-1}\mathbf{X}\right\} &= E\left\{\mathbf{Y}^t \sum_{i=1}^{n} \frac{1}{\lambda_i}\mathbf{q}_i\mathbf{q}_i^t\mathbf{Y}\right\} \\ &= \sum_{i=1}^{n} \frac{1}{\lambda_i} E\left\{\mathbf{Y}^t\mathbf{q}_i\mathbf{q}_i^t\mathbf{Y}\right\} \end{aligned}$$

Since $\mathbf{Y}^t\mathbf{q}_i$ is a 1×1 matrix, a number, we have

$$\begin{aligned}
E\left\{\mathbf{Y}^t R^{-1}\mathbf{X}\right\} &= \sum_{i=1}^{n} \frac{1}{\lambda_i} E\left\{\mathbf{q}_i^t \mathbf{Y}\mathbf{Y}^t \mathbf{q}_i\right\} \\
&= \sum_{i=1}^{n} \frac{1}{\lambda_i} \mathbf{q}_i^t E\left\{\mathbf{Y}\mathbf{Y}^t\right\} \mathbf{q}_i \\
&= \sum_{i=1}^{n} \frac{1}{\lambda_i} \mathbf{q}_i^t (\mathbf{R}\mathbf{q}_i) \\
&= \sum_{i=1}^{n} \frac{1}{\lambda_i} \mathbf{q}_i^t \lambda_i \mathbf{q}_i \\
&= \sum_{i=1}^{n} \mathbf{q}_i^t \mathbf{q}_i \\
&= n
\end{aligned}$$

12. Random process $v_1[n]$ and $v_2[n]$ are independent and have the same correlation function

$$\mathbf{R}_v[n_1, n_0] = 0.5\delta[n_1 - n_0]$$

- What is the correlation function of the random process

$$x[n] = v_1[n] + 2v_1[n+1] + 3v_2[n-1]?$$

- Is this random process wide-sense stationary?
- Find the correlation matrix for a random vector consisting of eight consecutive samples of x[n]

Answer: (1)

$$\begin{aligned}
R_k &= E\{x[n]x[n-k]\} \\
&= E\{(v_1[n] + 2v_1[n+1] + 3v_2[n-1])(v_1[n-k] + 2v_1[n-k+1] + 3v_2[n-k-1])\} \\
&= 7\delta[k] + \delta[k+1] + \delta[k-1] + 18E\{v_1[n]\}E\{v_2[n]\}
\end{aligned}$$

(2)

$$E\{x[n]\} = 3E\{v_1[n]\} + 3E\{v_2[n]\}$$

If we assume $v_1[n]$ and $v_2[n]$ are stationary, then $E\{x[n]\}$ does not change with time.
R_k is a function only of the time difference k.
So, $x[n]$ is wide-sense stationary.

(3) Omitted.

13. Find an expression for the power spectral density of a continuous real random process with correlation function

$$R_{x_c}^c(\tau) = \sigma_x^2 e^{-\frac{|\tau|}{\tau_0}}$$

Answer:

$$\begin{aligned} S_x(\omega) &= \int_{-\infty}^{\infty} \sigma_x^2 e^{-\frac{|\tau|}{\tau_0}} e^{-j\omega\tau} d\tau \\ &= \sigma_x^2 \int_{-\infty}^{0} e^{\frac{\tau}{\tau_0}} e^{-j\omega\tau} d\tau + \sigma_x^2 \int_{0}^{\infty} e^{-\frac{\tau}{\tau_0}} e^{-j\omega\tau} d\tau \\ &= \frac{\sigma_x^2}{\frac{1}{\tau_0} - j\omega} + \frac{\sigma_x^2}{\frac{1}{\tau_0} + j\omega} \\ &= \frac{2\tau_0}{1 + (\omega\tau_0)^2} \end{aligned}$$

14.

- Determine the mean of the exponential density function $f_x(X) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & otherwise \end{cases}$ and express the density in terms of the mean parameter $\mu = E\{x\}$
- Assume that you are given N independent samples $x_1, x_2, ..., x_N$ of the random variable x. What is the maximum likelihood estimate for the mean μ?
- Is this estimate unbiased?
- Is it consistent?
- What is the variance of the estimate? Is this a minimum-variance estimate?

Answer:

(1)

$$\mu = E[x] = \int_0^{\infty} \alpha e^{-\alpha x} dx = \frac{1}{\alpha}$$

(2)

$$\begin{aligned} f_{x_1,x_2,\cdots,x_N}(x_1, x_2, \cdots, x_N) &= \prod_{i=1}^{N} f_{x_i}(x_i) \\ &= \alpha^N e^{-\alpha \sum_{i=1}^{N} x_i} \end{aligned}$$

$$\frac{df_{x_1,x_2,\cdots,x_N}(x_1, x_2, \cdots, x_N)}{d\alpha} = N\alpha^{N-1} e^{-\sum_{i=1}^{N} \alpha} + \alpha^N \left(-\sum_{i=1}^{N} x_i \right) e^{-\sum_{i=1}^{N} \alpha} = 0$$

$$\Downarrow$$

$$\alpha = \frac{N}{\sum_{i=1}^{N} x_i}$$

So, the maximum likelihood estimate of μ is

$$\hat{\mu} = \frac{\sum_{i=1}^{N} x_i}{N}$$

(3)

$$\begin{aligned}
E[\hat{\mu}] &= E\left[\frac{\sum_{i=1}^{N} x_i}{N}\right] \\
&= \frac{\sum_{i=1}^{N} E[x_i]}{N} \\
&= \frac{\sum_{i=1}^{N} \mu}{N} \\
&= \mu
\end{aligned}$$

So, the estimate is unbiased.
(4)

$$\begin{aligned}
E[\hat{\mu}_N^2] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)\left(\frac{1}{N}\sum_{j=1}^{N} x_j\right)\right] \\
&= \frac{1}{N^2}\left(\sum_{i=1}^{N} E[x_i^2] + \sum_{i\neq j} E[x_i]E[x_j]\right) \\
&= \frac{1}{N^2}\left(\sum_{i=1}^{N} \frac{2}{\alpha^2} + N(N-1)\frac{1}{\alpha^2}\right) \\
&= \frac{1}{\alpha^2}(1+\frac{1}{N})
\end{aligned}$$

The variance of the estimate is

$$\begin{aligned}
var(\hat{\mu}_N) &= E[\hat{\mu}^2] - E[\hat{\mu}]^2 \\
&= \frac{1}{\alpha^2}(1+\frac{1}{N}) - \frac{1}{\alpha^2} \\
&= \frac{1}{N\alpha^2} \\
&= \frac{\mu^2}{N}
\end{aligned}$$

$$var(\hat{\mu}_N) < var(\hat{\mu}_{N-1})$$

Also, from (3), $\hat{\mu_N}$ is unbiased.
So, $\hat{\mu_N}$ is a consistent estimate.
(5)

$$f_{\mathbf{x}|\mu}(\mathbf{x}|\mu) = \frac{1}{\mu^N} e^{-\frac{\sum_{i=1}^{N} x_i}{\mu}}$$

$$\ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)] = -N\ln\mu - \frac{\sum_{i=1}^{N} x_i}{\mu}$$

$$\frac{\partial^2}{\partial\mu^2}\ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)] = \frac{N}{\mu^2} - \frac{2\sum_{i=1}^{N} x_i}{\mu^3}$$

$$E\left[\frac{\partial^2}{\partial\mu^2}\ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)]\right] = -\frac{N}{\mu^2}$$

$$\left(-E\left[\frac{\partial^2}{\partial\mu^2}\ln[f_{\mathbf{x}|\mu}(\mathbf{x}|\mu)]\right]\right)^{-1} = \frac{\mu^2}{N} = var(\hat{\mu}_N)$$

That is, the estimate $\hat{\mu}_N = \frac{\sum_{i=1}^{N} x_i}{N}$ hits the Cramer-Rao Bound.
So, the unbiased estimate $\hat{\mu}_N$ is a minimum variance estimate.