1. Let
$$\mathbf{R} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$
Express $\mathbf{R}$ as $\mathbf{R} = \mathbf{Q}\mathbf{\Omega}\mathbf{Q}^H$, where $\mathbf{\Omega}$ is diagonal.

2. The two-dimensional covariance matrix can be expressed as:
$$\mathbf{C} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho^*\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$
   (a) Find the simplest expression for the eigenvalues of $\mathbf{C}$.
   (b) Specialize the results to the case $\sigma^2 = \sigma_2^2 = \sigma_2^2$.
   (c) What are the eigenvectors in the special case (b) when ρ is real?

3. Let
$$x[n] = Ae^{j\omega_0 n}$$
where the complex amplitude A is a RV with random magnitude and phase
$$A = |A|e^{j\phi}.$$
Show that a sufficient condition for the random process to be stationary is that the amplitude and phase are independent and that the phase is uniformly distributed over $[-\pi, \pi]$.

4. Let X_i be i.i.d. RVs uniformly distributed on $[0, 1]$ and define
$$Y = \sum_{i=1}^{20} X_i.$$
Utilize Tchebycheff's inequality to determine a bound for $Pr\{8 < Y < 12\}$.

5. Let $X \sim \mathcal{N}(0, 2\sigma^2)$ and $Y \sim \mathcal{N}(0, \sigma^2)$ be independent RVs. Also, define $Z = XY$. Find the Bays estimate of X from observation Z:
   (a) Using the squared error criteria.
   (b) Using the absolute error criteria.

6. Let X and Y be independent RVs characterized by $f_X(x) = ae^{-ax}U(x)$ and $f_Y(y) = ae^{-ay}U(y)$. Also, define $Z = XY$. Find the Bays estimate of X from observation Z using the uniform cost function.

7. Random processes $x[n]$ and $y[n]$ are defined by
$$x[n] = v_1[n] + 3v_2[n-1]$$
$$y[n] = v_2[n+1] + 3v_2[n-1]$$
where $v_1[n]$ and $v_2[n]$ are independent white noise processes, each with variance 0.5.
   (a) Determine the autocorrelation functions of x and y. Are the processes WSS?

(b) Determine the cross–correlation functions between x and y. Are the processes jointly WSS?

8. A random process $x(n)$ has correlation matrix

$$\mathbf{R} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) Determine the KLT basis functions.

(b) Determine the minimum mean square error achievable when using a single term approximation, e.g.,

$$\hat{x}(n) = c_1(n)\mathbf{q}_1$$

(c) Determine the error when 2, 3 and 4 terms are used.

9. Let

$$v[n] = a_1^* x[n] + a_2^* y[n]$$

where x and y are WSS processes. Show that the coherence satisfies

$$|\Gamma(\omega)|^2 = \frac{|S_{xy}(\omega)|^2}{S_x(\omega)S_y(\omega)} \geq 1$$

where $S_x(\omega), S_y(\omega), S_{xy}(\omega)$ are PSD and cross–PSD functions. HINT: Write the output PSD function using a matrix expression, $S_v(\omega) = \mathbf{a}^H \boldsymbol{S}\mathbf{a}$, and use known properties.

10. Let $\{x[n]\}$ be a process of i.i.d. RVs uniform on $[-1, 1]$. The process is passed through a LTI system with impulse response $h[n] = \left(\frac{1}{2}\right)^2 U[n]$, yielding output $y[n]$.

(a) Determine $R_{yx}[l]$.

(b) Determine $R_y[l]$.

(c) Determine $S_y(\omega)$

11. (a) Determine the mean of the exponential density function $f_x(x) = \alpha e^{-\alpha x} U(x)$, and expressed the density in terms of the mean parameter $\mu = E\{x\}$.

(b) Given independence samples $x_1, x_2, \ldots, x_N$, determine the ML estimate of μ.

(c) Is the estimate unbiased?

(d) Is it consistent?

(e) What is the variance of the estimate? Is it a minimum variance estimate?

12. The joint density function of random variables x and y is given by

$$f_{xy}(x, y) \begin{cases} 6x & 0 \leq x \leq 1; 0 \leq y \leq 1 - x \\ 0 & \text{otherwise} \end{cases}$$

(a) Determined and sketch the conditional density function $f_{y|x}(y|x)$.

(b) Determine the MAP estimate of y.

(c) Determine the MS estimate of y.

(d) Determine the MAE estimate of y.

13. A two-dimensional vector $\mathbf{x}$ and a random variable y have the joint density

$$f_{\mathbf{x}y}(\mathbf{x}, y) \begin{cases} (y + 3x_1)x_2 & 0 \leq x_1, x_2, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $\mathbf{x}$ represents the observation.

(a) Determine the MAP estimate of y.

(b) Determine the MS estimate of y.

(c) Determine the MAE estimate of y.

14. A random process $x[n]$ is generated according to the difference equation

$$x[n] = \rho x[n-1] + \eta[n]$$

where ρ is a constant and is a binary whitenoise sequence taking on values -1 and $+1$ with equal probabilities.

(a) Generate and plot $M = 50$ samples of the random sequence for $\rho = 0.95, 0.70$, and -0.95. What differences do you observe in these three random sequences?

(b) Repeat the above with $\eta[n]$ a white noise Gaussian sequence with unit variance.

(c) Let $\hat{R}_x[l]$ be the sample autocorrelation. Define the estimated correlation coefficient $\hat{\rho}$ as

$$\hat{\rho} = \frac{\hat{R}_x[1]}{\hat{R}_x[0]}$$

Compute $\hat{\rho}$ for each of the Gaussian noise driven sequences and for several sequence lengths. How well does the estimated value compare with the theoretical value?

(d) Plot the estimated $\hat{R}_x[l]$ and true $R_x[l]$ autocorrelation functions for $0 \leq l \leq 1$ for each of the Gaussian noise driven sequences.

(e) What happens if $|\rho| > 1$?