1. A random process $x(n)$ has correlation matrix

$$\mathbf{R} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   (a) Determine the KLT basis functions.

   (b) Determine the minimum mean square error achievable when using a single term approximation, e.g.,

$$\hat{x}(n) = c_1(n)\mathbf{q}_1$$

   (c) Determine the error when 2, 3 and 4 terms are used.

2. A real random process is defined by

$$x(n) = A\cos\omega_o n + w(n)$$

   where A is a Gaussian random variable with mean zero and variance σ_A^2 and $w(n)$ is a white noise process independent of A with variance σ_w^2.

   (a) What is the correlation function of $x(n)$?

   (b) Can the power spectral density of $x(n)$ be defined? If so, what is the power spectral density function?

   (c) Repeat (a) and (b) in the case when the cosine has an independent random phase uniformly distributed on $[-\pi, \pi]$.

3. A linear system is defined by

$$y(n) = 0.7y(n-1) + x(n) - x(n-1)$$

   (a) Compute the first four values of $R_{yx}(l)$ it is known that $R_x(l) = \delta(l)$.

   (b) What is $R_{yx}(l)$ for $-3 \leq l \leq 3$?

   (c) What is the power spectral density function $S_y(\omega)$?

4. A random process is defined by

$$x(n) = s(n) + \eta(n)$$

   where $\eta(n)$ is a unit variance white noise process and $s(n)$ is defined by

$$s(n) = \rho s(n-1) + w(n)$$

   where $w(n)$ is another unit variance white noise process independent of $\eta(n)$.

   (a) Determine the correlation function $R_x(l)$.

   (b) Determine the power spectral density function $S_x(\omega)$.

5. Recall the system

$$y(n) = \mathbf{w}^T \mathbf{x}(n)$$

where $\mathbf{x}(n) = \mathbf{u}(n) + \mathbf{v}(n)$. It was proven in class that the matched filter for a deterministic signal, corrupted by white noise, is given

$$\mathbf{w} = k\mathbf{u}^*(n)$$

where $\mathbf{w}$ is the vector of filter coefficients and $\mathbf{u}(n)$ is the deterministic signal. Suppose now the noise is colored (i.e.$\mathbf{R}_v \neq \sigma_v^2 \mathbf{I}$). Utilize the transformation

$$\mathbf{x}'(n) = k\mathbf{R}_v^{-1/2}\mathbf{x}(n)$$

to determine a similar result for the colored noise case.

6. (a) Determine the mean of the exponential density function $f_x(x) = \alpha e^{-\alpha x} U(x)$, and expressed the density in terms of the mean parameter $\mu = E\{x\}$.
   (b) Given independence samples $x_1, x_2, \ldots, x_N$, determine the ML estimate of μ.
   (c) Is the estimate unbiased?
   (d) Is it consistent?
   (e) What is the variance of the estimate? Is it a minimum variance estimate?

7. The joint density function of random variables x and y is given by

$$f_{xy}(x, y) \begin{cases} 8xy & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determined and sketch the conditional density function $f_{y|x}(y|x)$.
   (b) Determine the MAP estimate of y.
   (c) Determine the MS estimate of y.
   (d) Determine the MAE estimate of y.

8. A two-dimensional vector $\mathbf{x}$ and a random variable y have the joint density

$$f_{\mathbf{x}y}(\mathbf{x}, y) \begin{cases} A(y + 6x_1)x_2 & 0 \leq x_1, x_2, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $\mathbf{x}$ represents the observation.

   (a) Determine the MAP estimate of y.
   (b) Determine the MS estimate of y.
   (c) Determine the MAE estimate of y.

9. A random process $x[n]$ is generated according to the difference equation

$$x[n] = \rho x[n-1] + \eta[n]$$

where ρ is a constant and is a binary whitenoise sequence taking on values -1 and $+1$ with equal probabilities.

(a) Generate and plot $M = 50$ samples of the random sequence for $\rho = 0.95, 0.70$, and -0.95. What differences do you observe in these three random sequences?

(b) Repeat the above with $\eta[n]$ a white noise Gaussian sequence with unit variance.

(c) Let $\hat{R}_x[l]$ be the sample autocorrelation. Define the estimated correlation coefficient $\hat{\rho}$ as

$$\hat{\rho} = \frac{\hat{R}_x[1]}{\hat{R}_x[0]}$$

Compute $\hat{\rho}$ for each of the Gaussian noise driven sequences and for several sequence lengths. How well does the estimated value compare with the theoretical value?

(d) Plot the estimated $\hat{R}_x[l]$ and true $R_x[l]$ autocorrelation functions for $0 \leq l \leq 1$ for each of the Gaussian noise driven sequences.

(e) What happens if $|\rho| > 1$?