1. A signal with correlation function $R_x[l] = \rho^{|l|}$ is applied to a linear shift-invariant system with impulse response $h[n] = \delta[n] - \delta[n-1]$.

- Compute and plot the cross–correlation function $R_{yx}[l]$.
- Compute and plot the output correlation function $R_y[l]$.

2. In a matched filter application the the SNR is given by

$$SNR = \frac{\mathbf{w}^H \mathbf{s}^* \mathbf{s}^T \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}}$$

which reduces to

$$SNR = \frac{\mathbf{w}^H \mathbf{s}^* \mathbf{s}^T \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}}$$

when $\mathbf{R}_n = \sigma^2 \mathbf{I}$, yielding the solution $\mathbf{w} = k\mathbf{s}^*$. Show that for colored noise ($\mathbf{R}_n \neq \sigma^2 \mathbf{I}$) that using the transformation

$$\mathbf{x}' = \mathbf{R}_n^{-1/2} \mathbf{x}$$

reduces the problem to the white noise case with solution $\mathbf{w} = k\mathbf{R}^{-1}\mathbf{s}^*$.

3. A random variable x has the uniform density

$$f_x(X) = \begin{cases} 1/a & 0 \leq X \leq a \\ 1 & \text{otherwise} \end{cases}$$

- Determine the likelihood function for $N = 1$ and $N = 2$ and sketch it. Find the maximum likelihood estimate of the parameter a for these two cases.
- Determine the maximum likelihood estimate of the parameter a for arbitrary N.
- Show that the maximum likelihood estimate for the parameter a is a biased estimate for $N = 1$ and $N = 2$.
- Find the expected value of the maximum likelihood estimate of a for arbitrary N and show that the estimate is asymptotically unbiased.

4. A random variable x is given by $x = y + \eta$ where y is uniformly distributed over $[-\alpha, \alpha]$ and η is uniformly distributed over $[-\beta, \beta]$. Assume $|\alpha| < |\beta|$.

- Compute and sketch the density functions $f_{x|y}, f_{xy}, f_x$, and $f_{y|x}$.
- The random variable x is observed. If η is considered noise, what is the mean square estimate of y?
- What is the MAP estimate of y? Does a unique estimate exist?

5. A two-dimensional random vector $\mathbf{x}$ and a random variable y have the joint density function

$$f_{\mathbf{x},y}(\mathbf{X}, Y) = \begin{cases} (Y + 3X_1)X_2 & 0 \leq X_1, X_2, Y \leq 1 \\ 0 & \text{else} \end{cases}$$

Assume that $\mathbf{x}$ represents the observation.

- Find the mean–square estimate of y.
- Find the MAP estimate of y.

6. A random process $x[n]$ is generated according to the difference equation

$$x[n] = \rho x[n-1] + \eta[n]$$

where ρ is a constant and $\eta[n]$ is a binary white noise sequence taking on values -1 and $+1$ with equal probabilities.

- Generate and plot $M = 50$ samples of the random sequence for $\rho = 0.95, 0.70$, and -0.95. What differences do you observe in these three random sequences?
- Repeat the above with $\eta[n]$ a white noise Gaussian sequence with unit variance.
- Let $\hat{R}_x[l]$ be the sample autocorrelation. Define the estimated correlation coefficient $\hat{\rho}$ as

  $$\hat{\rho} = \frac{\hat{R}_x[1]}{\hat{R}_x[0]}$$

  Compute $\hat{\rho}$ for each of the Gaussian noise driven sequences and for several sequence lengths. How well does the estimated value compare with the theoretical value?
- Plot the estimated $\hat{R}_x[l]$ and true $R_x[l]$ autocorrelation functions for $0 \leq l \leq 1$ for each of the Gaussian noise driven sequences.
- What happens if $|\rho| > 1$?

7. Generate the KLT transform of an image. (You may select any 512×512 image or use one from the /usa/images/ directory.) Plot the reconstruction MSE vs. the number of basis functions used in the 5×5 and 7×7 observation window cases. For the 5×5 case, also show each of the image approximations.

8. We want to generate samples of a Guassian process with autocorrelation $r_x(l) = (\frac{1}{2})^{|l|} + (-\frac{1}{2})^{|l|}$ for all l. Find the difference equation that generates the process x(n) when excited by w(n)$\sim$ WGN(0,1)

9. Given the AR process x(n)=x(n-1)-0.5x(n-2)+w(n), complete the following tasks.

- Determine $\rho_x(1)$
- Using $\rho_x(0)$ and $\rho_x(1)$, compute $\{\rho_x(l)\}_2^{15}$ by the corresponding difference equation
- Plot $\rho_x(l)$ and use the resulting graph to estimate its period
- Compared the period obtained in part(c) with the value obtained using the PSD of the model (Hinit: Use the frequency of the PSD peak.)

10. Consider an AR(2) process x(n) with d_0=1, a_1=-1.6454, a_2=0.9025, and w(n)$\sim$WGN(0,1).

- Generate 100 samples of the process and use them to estimate the ACS$\hat{\rho}_x(l)$, using the following equation

  $$\hat{\rho}(l) = \frac{\sum_{n=l}^{N-1} x(n)x^*(n-l)}{\sum_{n=0}^{N-1} |x(n)|^2}$$

- Plot and compare the estimated and theretical ACS values for $0 \leq l \leq 10$
- Use the estimated value of $\rho_x(l)$ and the Yule-Walker equations to estimate the parameters of the model. Compare the estimated with the true values, and component on the accuracy of the approach.
- Use the estimated parameters to compute the PSDs of the process
- Compute and compare the estimated with the true PACS.

11. Show that if R is the correlation matrix of the random vector $\mathbf{X} : [\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}]$ and R^{-1} is its inverse, then $E\left\{\mathbf{X}R^{-1}\mathbf{X}^t\right\} = n$

12. Random process $v_1[n]$ and $v_2[n]$ are independent and have the same correlation function

$$\mathbf{R}_v[n_1, n_0] = 0.5\delta[n_1 - n_0]$$

- What is the correlation function of the random process

$$x[n] = v_1[n] + 2v_1[n+1] + 3v_2[n-1]?$$

- Is this random process wide-sense stationary?
- Find the correlation matrix for a random vector consisting of eight consecutive samples of x[n]

13. Find an expression for the power spectral density of a continuous real random process with correlation function

$$R^c_{x_c}(\tau) = \sigma_x^2 e^{-\frac{|\tau|}{\tau_0}}$$

14.

- Determine the mean of the exponential density function $f_x(X) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & otherwise \end{cases}$ and express the density in terms of the mean parameter $\mu = E\{x\}$
- Assume that you are given N independent samples $x_1, x_2, ..., x_N$ of the random variable x. What is the maximum likelihood estimate for the mean μ?
- Is this estimate unbiased?
- Is it consistent?
- What is the variance of the estimate? Is this a minimum-variance estimate?