1. A token is placed at the origin on a piece of graph paper. A coin biased to heads is given, $P(H) = 2/3$. If the result of a toss is heads, the token is moved one unit to the right, and if it is a tail the token is moved one unit to the left. Repeating this 1200 times, what is a probability that the token is on a unit N, where $350 \leq N \leq 450$? Simulate the system and plot the histogram using 10,000 realizations.

   Solution:
   Let $x =$ # of heads. Then $350 \leq x - (1200 - x) \leq 450 \Rightarrow 775 \leq x \leq 825$ and

   $$Pr(775x \leq 825) = \sum_{i=775}^{825} \binom{1200}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{1200-i}$$

   which can be approximated using the DeMoivre–Laplace approximation

   $$\sum_{i=i_1}^{i_2} \binom{n}{i} (p)^i (1-p)^{n-i} \approx \Phi\left(\frac{i_2 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{i_1 - np}{\sqrt{np(1-p)}}\right)$$

   where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-x^2/2} dx$

2. Random variable X is characterized by cdf $F_X(x) = (1 - e^{-x})U(x)$ and event C is defined by $C = \{0.5 < X \leq 1\}$. Determine and plot $F_X(x|C)$ and $f_X(x|C)$.

   Solution: Evaluating $Pr(X \leq x, 0.5 < X \leq 1)$ for the allowable three cases

   $$\begin{array}{ll} x < 0.5 & Pr(X \leq x, 0.5 < X \leq 1) = 0 \\ 0.5 \leq x \leq 1 & Pr(X \leq x, 0.5 < X \leq 1) = F_X(x) - F_X(0.5) = e^{-0.5} - e^{-x} \\ x > 1 & Pr(X \leq x, 0.5 < X \leq 1) = F_X(1) - F_X(0.5) = e^{-0.5} - e^{-1} = 0.2386 \end{array}$$

   Also, $Pr(C) = F_X(1) - F_X(0.5) = e^{-0.5} - e^{-1} = 0.2386$. Thus

   $$f_X(x|C) = \frac{Pr(X \leq x, 0.5 < X \leq 1)}{Pr(0.5 < X \leq 1)} = \begin{cases} 0 & x < 0.5 \\ (e^{-0.5} - e^{-x})/0.2386 & 0.5 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

3. Prove that the characteristic function for the univariate Gaussian distribution, $N(\eta, \sigma^2)$, is

   $$\phi(\omega) = \exp\left(j\omega\eta - \frac{\omega^2\sigma^2}{2}\right)$$

   Next determine the moment generating function and determine the first four moments.

Solution:

$$\begin{aligned}
\phi(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{(x-\eta)^2}{2\sigma^2}\right\} e^{j\omega x} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{(x^2 - 2\eta x + \eta^2 - 2j\omega x\sigma^2)}{2\sigma^2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{(x - (\eta x + j\omega\sigma^2)^2}{2\sigma^2}\right\} \exp\left\{\frac{(-\eta^2 + (\eta^2 + j\omega\sigma^2\eta)^2}{2\sigma^2}\right\} dx \\
&= \exp\left\{\frac{(-\eta^2 + (\eta^2 + j\omega\sigma^2\eta)^2}{2\sigma^2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{(x - (\eta x + j\omega\sigma^2)^2}{2\sigma^2}\right\} dx \\
&= \exp\left\{\frac{(-\eta^2 + (\eta^2 + j\omega\sigma^2\eta)^2}{2\sigma^2}\right\}
\end{aligned}$$

which reduces to $\phi(\omega) = \exp\left(j\omega\eta - \frac{\omega^2\sigma^2}{2}\right)$. The moment generating function is simple

$$\Phi(s) = \exp\left(s\eta + \frac{s^2\sigma^2}{2}\right)$$

and $m_k = \frac{d^k\Phi(s)}{d^k s}|_{s=0}$, which yields

$$m_1 = \eta \qquad m_2 = \sigma^2 + \eta^2$$
$$m_3 = 3\eta\sigma^2 + \eta^3 \qquad m_4 = 3\sigma^4 + 6\sigma^2\eta^2 + \eta^4$$

4. Let $Y = X^2$. Determine $f_Y(y)$ for:

   (a) $f_X(x) = 0.5\exp\{-|x|\}$
   (b) $f_X(x) = \exp\{-|x|\}U(X)$

   Solution: $Y = X^2 \Rightarrow X = \pm\sqrt{y}$ and $dY/dX = 2X$. Thus

$$f_Y(y) = \left.\frac{f_X(x)}{|2x|}\right|_{x=\sqrt{y}} + \left.\frac{f_X(x)}{|2x|}\right|_{x=-\sqrt{y}}$$

   Substituting and simplifying

$$f_X(x) = 0.5\exp\{-|x|\} \quad \Rightarrow \quad f_Y(y) = \frac{1}{2\sqrt{y}} e^{-\sqrt{y}} U(y)$$
$$f_X(x) = \exp\{-|x|\}U(x) \quad \Rightarrow \quad f_Y(y) = \frac{1}{2\sqrt{y}} e^{-\sqrt{y}} U(y)$$

5. Given the joint pdf $f_{XY}(x,y)$

$$f_{XY}(x,y) = \begin{cases} 8xy, & 0 < y < 1, 0 < x < y \\ 0, & \text{otherwise} \end{cases}$$

   Determine (a) $f_x(x)$, (b) $f_Y(y)$, (c) $f_Y(y|x)$, and (d) $E[Y|x]$.

   Solution:

(a) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_x^1 8xydy = \begin{cases} 4x - 4x^3 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_o^y 8xydx = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

(c) $f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} \frac{2y}{1-x^2} & x < y < 1 \\ 0 & \text{otherwise} \end{cases}$

(d) $E[Y|x] = \int_{-\infty}^{\infty} y f_Y(y|x)dy = \int_x^1 \frac{2y^2}{1-x^2}dy = \frac{2}{3}\left(\frac{1-x^3}{1-x^2}\right) = \frac{2}{3}\left(\frac{1+x+x^2}{1+x}\right)$

6. Let W and Z be RVs defined by

$$W = X^2 + Y^2 \qquad \text{and} \qquad Z = X^2$$

where X and Y are independent; $X, Y \sim N(0,1)$.

(a) Determine the joint pdf $f_{WZ}(w,z)$.

(b) Are W and Z independent?

Solution: Given the system of equations

$$J\begin{pmatrix} w & z \\ x & y \end{pmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & 0 \end{vmatrix} = 4|xy|$$

Note we must have $w, z \geq 0$ and $w \geq z$. Thus the inverse system (roots) are

$$x = \pm\sqrt{z}, \qquad y = \pm\sqrt{w-z}.$$

Thus

$$f_{WZ}(w,z) = \frac{f_{XY}(x,y)}{4|xy|}\Bigg|_{\substack{x=\pm\sqrt{z} \\ y=\pm\sqrt{w-z}}} \qquad (*)$$

Note also that, since $X, Y \sim N(0,1)$,

$$f_{XY}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}} \qquad (**)$$

Substituting $(**)$ into $(*)$ [which has four terms] and simplifying yields

$$f_{WZ}(w,z) = \frac{e^{w/2}}{2\pi\sqrt{z(w-z)}}U(w-z)U(z) \qquad (***)$$

Note W and Z are not independent. Counter example proof: Suppose W and Z *are* independent. Then $f_W(w)f_Z(z) > 0$ for all $w, z > 0$. But this violates $(***)$, as $f_{WZ}(w,z) > 0$ only for $w \geq z$.