1. Let $f_x(t)$ be symmetric about 0. Prove that μ is the expected value of a sample distributed according to $f_{x-\mu}(t)$.

   Solution.
   Since $f_x(t)$ is symmetric about 0, $f_x(t)$ is even.

$$\begin{aligned} E[(x-\mu)] &= \int_{-\infty}^{+\infty} t f_{x-\mu}(t)dt \\ &= \int_{-\infty}^{+\infty} t f_x(t-\mu)dt \end{aligned}$$

   Let $u = t - \mu$,

$$\begin{aligned} E[(x-\mu)] &= \int_{-\infty}^{+\infty} u + \mu f_x(u)du \\ &= \int_{-\infty}^{+\infty} \underbrace{u f_x(u)}_{odd} du + \int_{-\infty}^{+\infty} \mu f_x(u)du \\ &= 0 + \mu \int_{-\infty}^{+\infty} f_x(u)du \\ &= \mu \end{aligned}$$

2. The complimentary cumulative distribution function is defined as $Q_x(x) = 1 - F_x(x)$, or more explicitly in the zero mean, unit variance Gaussian distribution case as

$$Q_x(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt.$$

   Show that

$$Q_x(x) \approx \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{1}{2}x^2\right).$$

   Hint: use integration by parts on $Q_x(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}t} t \exp\left(-\frac{1}{2}t^2\right) dt$. Also explain why the approximation improves x as increases.

   Solution.
   Recall integration by parts: $\int_a^b f(t)g'(t)dt = f(t)g(t)|_a^b - \int_a^b f'(t)g(t)dt$. Let $g'(t) = t\exp\left(-\frac{1}{2}t^2\right)$ and $f(t) = \frac{1}{\sqrt{2\pi}t}$

$$Q_x(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}t} t \exp\left(-\frac{1}{2}t^2\right) dt$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{2\pi}t}\exp\left(-\frac{1}{2}t^2\right)\Big|_x^\infty - \underbrace{\int_x^\infty \frac{1}{\sqrt{2\pi}t^2}\exp\left(-\frac{1}{2}t^2\right)dt}_{\to 0 \text{ as } x\to\infty} \\
&\approx \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{1}{2}x^2\right)
\end{aligned}
$$

Since $\int_x^\infty \frac{1}{\sqrt{2\pi}t^2}\exp\left(-\frac{1}{2}t^2\right)dt$ goes to zero as x goes to infinity, the approximation improves x as increase.

3. The probability density function for a two dimensional random vector is defined by

$$
f_{\mathbf{x}}(\mathbf{x}) = \begin{cases} Ax_1^2x_2 & x_1, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

(a) Determine $F_{\mathbf{x}}(\mathbf{x})$ and the value of A.

(b) Determine the marginal density $f_{x_2}(x)$.

(c) Are $f_{x_1}(x)$ and $f_{x_2}(x)$ independent? Show why or why not.

Solution.

(a)

$$
\begin{aligned}
F_{x_1,x_2}(\infty,\infty) &= \int_0^1\int_0^{1-x_1} Ax_1^2x_2dx_2dx_1 \\
&= \int_0^1 Ax_1^2\frac{x_2^2}{2}\Big|_0^{x_2} \\
&= \int_0^1 Ax_1^2\frac{(1-x_1)^2}{2}dx_1 \\
&= \frac{A}{2}\int_0^1 (x_1^4 - 2x_1^3 + x_1^2)dx_1 \\
&= \frac{A}{60} \\
&= 1 \qquad (1)
\end{aligned}
$$

Therefore, A = 60. Defining $F_{x_1,x_2}(u,v) = Pr(x_1 \leq u, x_2 \leq v)$, we have

- $x_1 < 0$ or $x_2 < 0$, then $F(x_1, x_2) = 0$.

- $x_1, x_2 \geq 0$ and $x_1 + x_2 \leq 1$, then

$$\begin{aligned} F(x_1, x_2) &= \int_0^{x_1} \int_0^{x_2} 60u^2 v dv du \\ &= 10x_1^3 x_2^2 \end{aligned}$$

- $0 \leq x_1, x_2 \leq 1$ and $x_1 + x_2 \geq 1$, then

$$\begin{aligned} F(x_1, x_2) &= 1 - \int_0^{1-x_2} \int_{x_2}^{1-u} 60u^2 v dv du - \int_{x_1}^{1} \int_0^{1-u} 60u^2 v dv du \\ &= 10x_2^2 - 20x_2^3 + 15x_2^4 - 4x_2^5 + 10x_1^3 - 15x_1^4 + 6x_1^5 - 1 \end{aligned}$$

- $0 \leq x_1 \leq 1$ and $x_2 \geq 1$, then

$$\begin{aligned} F(x_1, x_2) &= 1 - \int_{x_1}^{1} \int_0^{1-u} 60u^2 v dv du \\ &= 10x_1^3 - 15x_1^4 + 6x_1^5 \end{aligned}$$

- $0 \leq x_2 \leq 1$ and $x_1 \geq 1$, then

$$\begin{aligned} F(x_1, x_2) &= 1 - \int_0^{1-x_2} \int_{x_2}^{1-u} 60u^2 v dv du \\ &= 10x_2^2 - 20x_2^3 + 15x_2^4 - 4x_2^5 \end{aligned}$$

- $x_1, x_2 \geq 1$, then $F(x_1, x_2) = 1$.

So

$$F(x_1, x_2) = \begin{cases} 0 & x_1 < 0 \textit{ or } x_2 < 0 \\ 10x_1^3 x_2^2 & x_1, x_2 \geq 0, x_1 + x_2 \leq 1 \\ 10x_2^2 - 20x_2^3 + 15x_2^4 - 4x_2^5 + 10x_1^3 - 15x_1^4 + 6x_1^5 - 1 & 0 \leq x_1, x_2 \leq 1, x_1 + x_2 \geq 1 \\ 10x_1^3 - 15x_1^4 + 6x_1^5 & 0 \leq x_1 \leq 1, x_2 \geq 1 \\ 10x_2^2 - 20x_2^3 + 15x_2^4 - 4x_2^5 & 0 \leq x_2 \leq 1, x_1 \geq 1 \\ 1 & x_1, x_2 \geq 1 \end{cases}$$

(b)

$$\begin{aligned} f_{x_2}(x_2) &= \int_0^{1-x_2} 60x_1^2 x_2 dx_1 \\ &= 20x_2(1 - x_2)^3 \end{aligned}$$

(c) Since

$$\begin{aligned} f_{x_1}(x_1) &= \int_0^{1-x_1} 60x_1^2 x_2 dx_2 \\ &= 30x_1^2(1-x_1)^2 \end{aligned}$$

, $f_{x_1,x_2}(x_1,x_2) \neq f_{x_1}(x_1) f_{x_2}(x_2)$. Therefore, $f_{x_1}(x_1)$ and $f_{x_2}(x_2)$ are NOT independent.

4. Consider the two independent marginal distributions

$$f_{x_1}(x) = \begin{cases} 1 & 0 \le x_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{x_2}(x) = \begin{cases} 2x & 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event $x_1 \le x_2$.

   (a) Find and sketch $f_{\mathbf{x}}(\mathbf{x})$.

   (b) Determine $Pr\{A\}$.

   (c) Determine $f_{\mathbf{x}|A}(\mathbf{x}|A)$. Are the components independent, i.e., are $f_{x_1|A}(x|A)$ and $f_{x_2|A}(x|A)$ independent?

Solution.
(a) Since two marginal distributions are independent,

$$\begin{aligned} f_X(X) &= f_{x_1}(x_1) f_{x_2}(x_2) \\ &= \begin{cases} 2x_2 & 0 \le x_1, x_2 \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} Pr(A) &= \int_0^1 \int_0^{1-x_2} 2x_2 dx_1 dx_2 \\ &= \int_0^1 2x_2^2 dx_2 \\ &= \frac{2x_2^3}{3}\Big|_0^1 \\ &= \frac{2}{3} \end{aligned} \tag{2}$$

(c)

$$\begin{aligned} f_{X|A}(X|A) &= \frac{f_X(X)}{Pr(A)} \\ &= \begin{cases} 3x_2 & 0 \le x_1 < x_2 \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} f_{x_1|A}(x_1|A) &= \int_{x_1}^{1} 3x_2 dx_2 \\ &= \frac{3x_2^2}{2}\Big|_{x_1}^{1} \\ &= \frac{3(1-x_1)^2}{2}, \quad 0 \le x_1 \le 1 \end{aligned}$$

$$\begin{aligned} f_{x_2|A}(x_2|A) &= \int_{0}^{x_2} 2x_2 dx_1 \\ &= 2x_2^2, \quad 0 \le x_2 \le 1 \end{aligned}$$

$f_{X|A}(X|A) \neq f_{x_1|A}(x_1|A) f_{x_2|A}(x_2|A)$. Therefore, $f_{x_1|A}(x_1|A)$ and $f_{x_2|A}(x_2|A)$ are NOT independent.

5. The entropy $\mathcal{H}$ for a random vector is defined as $-E\{\ln f_{\mathbf{x}}(\mathbf{x})\}$. Show that for the complex Gaussian case

$$\mathcal{H} = N(1 + \ln \pi) + \ln |\mathbf{C}_x|.$$

Determine the corresponding expression when the vector is real.

Solution.

The complex Gaussian p.d.f. is

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\pi^N |\mathbf{C}_{\mathbf{x}}|} \exp[-(\mathbf{x} - m_{\mathbf{x}})^H \mathbf{C}_{\mathbf{x}}^{-1} (\mathbf{x} - m_{\mathbf{x}})]$$

Then,

$$\begin{aligned} \mathcal{H} &= -E\{\ln f_{\mathbf{x}}(\mathbf{x})\} \\ &= E[(\mathbf{x} - m_{\mathbf{x}})^H \mathbf{C}_{\mathbf{x}}^{-1} (\mathbf{x} - m_{\mathbf{x}})] + N \ln \pi + \ln |\mathbf{C}_{\mathbf{x}}| \end{aligned}$$

Note

$$
\begin{aligned}
E[(\mathbf{x}-m_{\mathbf{x}})^H\mathbf{C}_{\mathbf{x}}^{-1}(\mathbf{x}-m_{\mathbf{x}})] &= E[\text{trace}((\mathbf{x}-m_{\mathbf{x}})^H\mathbf{C}_{\mathbf{x}}^{-1}(\mathbf{x}-m_{\mathbf{x}}))] \\
&= \text{trace}(\mathbf{C}_{\mathbf{x}}^{-1}E[(\mathbf{x}-m_{\mathbf{x}})(\mathbf{x}-m_{\mathbf{x}})^H]) \\
&= \text{trace}(\mathbf{C}_{\mathbf{x}}^{-1}\mathbf{C}_{\mathbf{x}}) \\
&= \text{trace}(\mathbf{I}) = N
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathcal{H} &= N + N\ln\pi + \ln|\mathbf{C}_{\mathbf{x}}| \\
&= N(1+\ln\pi) + \ln|\mathbf{C}_{\mathbf{x}}|
\end{aligned}
$$

Similarly, when the vector is real

$$
\mathcal{H} = \frac{1}{2}N(1+\ln(2\pi)) + \frac{1}{2}\ln|\mathbf{C}_{\mathbf{x}}|
$$

6. Let

$$
\begin{aligned}
x &= 3u - 4v \\
y &= 2u + v
\end{aligned}
$$

where u and v are unit mean, unit variance, uncorrelated Gaussian random variables.

(a) Determine the means and variances of x and y.

(b) Determine the joint density of x and y.

(c) Determine the conditional density of y given x.

Solution.
(a)

$$
\begin{aligned}
E(x) &= E(3u-4v) \\
&= 3E(u) - 4E(v) \\
&= 3 - 4 \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
E(y) &= E(2u+v) \\
&= 2E(u) + E(v) \\
&= 2 + 1 \\
&= 3
\end{aligned}
$$

$$\begin{aligned}\sigma_x^2 &= E(x^2) - E^2(x) \\ &= E[(3u-4v)^2] - 1 \\ &= 25\end{aligned}$$

$$\begin{aligned}\sigma_y^2 &= E(y^2) - E^2(y) \\ &= E[(2u+v)^2] - 9 \\ &= 5\end{aligned}$$

(b) Note

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} u \\ v \end{bmatrix}$$

Thus

$$\mathbf{A}^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

and

$$\begin{aligned}f_{x,y}(x,y) &= \frac{f_{u,v}(\mathbf{A}^{-1}[x,y]^T)}{\text{abs } |\mathbf{A}|} \\ &= \frac{1}{11} f_{u,v}((x+4y)/11, (-2x+3y)/11) \\ &= \frac{1}{22\pi}\exp(-\frac{1}{2}[(\frac{x+4y}{11} - 1)^2 + (\frac{-2x+3y}{11} - 1)^2])\end{aligned}$$

(c) Note x is Gaussian

$$f_x(x) = \frac{1}{\sqrt{2\pi}\times 5}\exp\left(-\frac{1}{2\times 25}(x+1)^2\right)$$

Thus

$$\begin{aligned}f_{y|x}(y|x) &= \frac{f_{x,y}(x,y)}{f_x(x)} \\ &= \frac{\sqrt{2\pi}\times 5}{22\pi}\exp\left(-\frac{1}{2}[(\frac{x+4y}{11} - 1)^2 + (\frac{-2x+3y}{11} - 1)^2] + \frac{1}{2\times 25}(x+1)^2\right) \\ &= \frac{5}{22}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2}[(\frac{x+4y}{11} - 1)^2 + (\frac{-2x+3y}{11} - 1)^2 - \frac{1}{25}(x+1)^2]\right)\end{aligned}$$

7. Consider the orthogonal transformation of the correlated zero mean random variables x_1 and x_2

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note $E\{x_1^2\} = \sigma_1^2$, $E\{x_2^2\} = \sigma_2^2$, and $E\{x_1 x_2\} = \rho\sigma_1\sigma_2$. Determine the angle θ such that y_1 and y_2 are uncorrelated.

Solution.

$$\begin{cases} y_1 &= x_1\cos\theta + x_2\sin\theta \\ y_2 &= -x_1\sin\theta + x_2\cos\theta \end{cases}$$

$$\begin{aligned} E(y_1y_2) &= E[(x_1\cos\theta + x_2\sin\theta)(-x_1\sin\theta + x_2\cos\theta)] \\ &= \sin\theta\cos\theta E[x_2^2] + (\cos^2\theta - \sin^2\theta)E[x_1x_2] - \sin\theta\cos\theta E[x_1^2] \\ &= \sin\theta\cos\theta(\sigma_2^2 - \sigma_1^2) + (\cos^2\theta - \sin^2\theta)\rho\sigma_1\sigma_2 \\ &= \sin 2\theta \cdot \frac{(\sigma_2^2 - \sigma_1^2)}{2} + \cos 2\theta \cdot \rho\sigma_1\sigma_2 \end{aligned}$$

If y_1 and y_2 are uncorrelated, $E(y_1y_2) = 0$. For $-\pi/2 \le \theta < \pi/2$,

$$\theta = \frac{1}{2}\arctan\frac{2\rho\sigma_1\sigma_2}{\sigma_2^2 - \sigma_1^2}$$

8. The covariance matrix and mean vector for a real Gaussian density are

$$\mathbf{C}_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

and

$$\mathbf{m}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Determine the eigenvalues and eigenvectors.

(b) Generate a mesh plot of the distribution using MATLAB.

(c) Change the off-diagonal values to -0.5 and repeat (a) and (b).

Solution.

(a) Solve $|\mathbf{C_x} - \lambda \mathbf{I}| = 0$.

$$(1-\lambda)^2 - 0.25 = (\lambda - 0.5)(\lambda - 1.5) = 0$$

Hence, eigenvalues are 0.5 and 1.5. For $\lambda = 0.5$, the corresponding eigenvector is $[1, -1]^T$. For $\lambda = 1.5$, the corresponding eigenvector is $[1, 1]^T$.

(c) Eigenvalues are 0.5 and 1.5. For $\lambda = 0.5$, the corresponding eigenvector is $[1, 1]^T$. For $\lambda = 1.5$, the corresponding eigenvector is $[1, -1]^T$.

9. Let $\{x_k(n)\}_{k=1}^{K}$ be i.i.d. zero mean, unit variance uniformly distributed random variables and set

$$y_K(n) = \sum_{k=1}^{K} x_k(n).$$

   (a) Determine and plot the pdf of $y_K(n)$ for $K = 2, 3, 4$.
   (b) Compare the pdf's to the Gaussian density.
   (c) Perform the comparison experimentally using MATLAB. That is, generate K sequences of $n = 1, 2, \ldots, N$ uniformly distributed samples. Add the sequences and plot the resulting distribution (histogram). Fit the results to a Gaussian distribution for various K and N.

Solution.

(a) $\{x_k(n)\}_{k=1}^{K}$ are i.i.d. zero mean, unit variance uniformly distributed random variables.

$$f_{x_k}(x_k) = \begin{cases} 1/2a & x_k \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

Since $E[x_k^2] = 1$,

$$\begin{aligned} E[x_k^2] &= \frac{1}{2a}\int_{-a}^{a} x^2 dx \\ &= \frac{x^3}{6a}\Big|_{-a}^{a} \\ &= \frac{a^2}{3} \\ &= 1 \end{aligned}$$

$$\Rightarrow a = \sqrt{3}$$

That is

$$f_{x_k}(x_k) = \begin{cases} \frac{1}{2\sqrt{3}} & x_k \in [-\sqrt{3}, \sqrt{3}] \\ 0 & \text{otherwise} \end{cases}$$

For K=2, $y_2(n) = x_1(n) + x_2(n)$.

$$\begin{aligned} f_{y_2}(x) &= f_{x_1}(x) * f_{x_2}(x) \\ &= \begin{cases} \frac{x}{12} + \frac{1}{2\sqrt{3}} & -2\sqrt{3} \le x < 0 \\ -\frac{x}{12} + \frac{1}{2\sqrt{3}} & 0 < x \le 2\sqrt{3} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For K=3, $y_3(n) = x_1(n) + x_2(n) + x_3(n) = y_2(n) + x_3(n)$.

$$\begin{aligned} f_{y_3}(x) &= f_{y_2}(x) * f_{x_3}(x) \\ &= \begin{cases} \frac{(x+3\sqrt{3})^2}{48\sqrt{3}} & -3\sqrt{3} \le x < -\sqrt{3} \\ \frac{3-x^2}{8\sqrt{3}} & -\sqrt{3} \le x < \sqrt{3} \\ \frac{(x-3\sqrt{3})^2}{48\sqrt{3}} & \sqrt{3} \le x \le 3\sqrt{3} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$