1. Show that if $\beta_x(t) = f_x(t|\mathbf{x} > t)$, $\beta_y(t) = f_y(t|\mathbf{y} > t)$ and $\beta_x(t) = k\beta_y(t)$ then $1 - F_x(x) = [1 - F_y(x)]^k)$

Answer:

$$\begin{aligned} f_x(t|\mathbf{x} > t)dt &= P(t < \mathbf{x} < t + dt|\mathbf{x} > t) \\ &= \frac{P(t < \mathbf{x} < t + dt, \mathbf{x} > t)}{P(\mathbf{x} > t)} \\ &= \frac{f_x(t)dt}{1 - F_x(t)} \end{aligned}$$

$$\Rightarrow f_x(t|\mathbf{x} > t) = \frac{f_x(t)}{1 - F_x(t)}$$

In the same way

$$f_y(t|\mathbf{y} > t) = \frac{f_y(t)}{1 - F_y(t)}$$

Then

$$\begin{aligned} &\beta_x(t) = k\beta_y(t) \\ &\Rightarrow \tfrac{f_x(t)dt}{1-F_x(t)} = \tfrac{kf_y(t)}{1-F_y(t)} \\ &\Rightarrow \int_{-\infty}^{\infty} \tfrac{f_x(t)dt}{1-F_x(t)}dt = \int_{-\infty}^{\infty} \tfrac{kf_y(t)}{1-F_y(t)}dt \\ &\Rightarrow \int_{-\infty}^{\infty} \tfrac{dF_x(t)dt}{1-F_x(t)} = k\int_{-\infty}^{\infty} \tfrac{dF_y(t)}{1-F_y(t)} \\ &\Rightarrow \ln[1 - F_x(x)] = k\ln[1 - F(y)] \\ &\Rightarrow 1 - F_x(x) = [1 - F_y(x)]^k \end{aligned}$$

2. Express the density $f_x(y)$ of the RV $\mathbf{y} = g(\mathbf{x})$ in terms of $f_x(x)$ if (a) $g(x) = |x|$; (b) $g(x) = e^{-x}U(x)$.

Answer:
(a)

$$\begin{aligned} F_y(y) &= P(\mathbf{y} \le y) \\ &= P(g(\mathbf{x}) \le y) \\ &= P(|\mathbf{x}| \le y) \\ &= \begin{cases} 0 & y < 0 \\ P(-y \le \mathbf{x} \le y) & y \ge 0 \end{cases} \\ &= \begin{cases} 0 & y < 0 \\ P(\mathbf{x} \le y) - P(\mathbf{x} \le -y) & y \ge 0 \end{cases} \\ &= \begin{cases} 0 & y < 0 \\ F_x(y) - F_x(-y) & y \ge 0 \end{cases} \end{aligned}$$

$$f_y(y) = \frac{dF_y(y)}{y}$$

$$
\begin{aligned}
&= \begin{cases} 0 & y < 0 \\ \frac{d[F_x(y)-F_x(-y)]}{dy} & y \geq 0 \end{cases} \\
&= \begin{cases} 0 & y < 0 \\ \frac{dF_x(y)}{dy} - \frac{dF_x(-y)}{dy} & y \geq 0 \end{cases} \\
&= \begin{cases} 0 & y < 0 \\ f_x(y) + f_x(-y) & y \geq 0 \end{cases}
\end{aligned}
$$

(b)

$$
\begin{aligned}
F_y(y) &= P(\mathbf{y} \leq y) \\
&= P(g(\mathbf{x}) \leq y) \\
&= P(e^{-\mathbf{x}}U(\mathbf{x}) \leq y) \\
&= \begin{cases} 0 & y < 0 \\ P(e^{-\mathbf{x}} \leq y) & y \geq 0 \end{cases} \\
&= \begin{cases} 0 & y < 0 \\ P(\mathbf{x} \geq -\ln(y)) & y \geq 0 \end{cases} \\
&= \begin{cases} 0 & y < 0 \\ 1 - F_x(-\ln(y)) & y \geq 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_y(y) &= \frac{F_y(y)}{y} \\
&= \begin{cases} 0 & y < 0 \\ \frac{d[1-F_x(-\ln(y))]}{dy} & y \geq 0 \end{cases} \\
&= \begin{cases} 0 & y < 0 \\ \frac{f_x(-\ln(y))}{y} & y \geq 0 \end{cases}
\end{aligned}
$$

3. The RVs $\mathbf{x}$ and $\mathbf{y}$ are independent with exponential densities
   $f_x(x) = \alpha e^{-\alpha x}U(x), f_y(y) = \beta e^{-\beta y}U(y)$
   Find the densities of the following RVs: $\mathbf{2x} + \mathbf{y}$; $\frac{\mathbf{x}}{\mathbf{y}}$

   Answer: (1) Let $\mathbf{z} = \mathbf{w} + \mathbf{y}$ and $\mathbf{w} = 2\mathbf{x}$

$$
f_w(w) = \frac{1}{2}f_x(\frac{w}{2}) = \frac{\alpha}{2}e^{-\frac{\alpha}{2}w}U(w)
$$

Since w and y are independent, we have

$$
\begin{aligned}
f_z(z) &= f_w(z) * f_y(z) \\
&= \int_{-\infty}^{\infty} f_y(z-w)f_w(w)dw \\
&= \int_0^z \beta e^{-\beta(z-w)}\frac{\alpha}{2}e^{-\frac{\alpha}{2}w}dw \\
&= \frac{\alpha\beta}{2\beta - \alpha}(e^{-\frac{\alpha z}{2}} - e^{-\beta z})U(z)
\end{aligned}
$$

(2)

$$\begin{aligned} F_z(z) &= P(\mathbf{z} \leq z) = P(\mathbf{x} \leq z\mathbf{y}) \\ &= \int_{-\infty}^{\infty} P(\mathbf{x} \leq z\mathbf{y} | \mathbf{y} = y) f_y(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{zy} f_x(x) dx f_y dy \\ &= \int_{0}^{\infty} \int_{0}^{zy} \alpha e^{-\alpha x} dx \beta e^{-\beta y} dy \\ &= \beta \int_{0}^{\infty} (e^{-\beta y} - e^{-(\alpha z+\beta)} y) dy \\ &= 1 - \frac{\beta}{\alpha z + \beta} \end{aligned}$$

$$f_z(z) = \frac{dF_z(z)}{dz} = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. The RVs $\mathbf{x}$ and $\mathbf{y}$ are $N(0;\sigma)$ and independent. Show that, if $\mathbf{z} = |\mathbf{x} - \mathbf{y}|$, then

$$E\{\mathbf{z}\} = 2\sigma/\sqrt{\pi}, E\{\mathbf{z}^2\} = 2\sigma^2.$$

Answer:

Let $\mathbf{z} = |\mathbf{w}|$, and $\mathbf{w} = \mathbf{x} - \mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{y}$ are Gaussian, so $\mathbf{w}$ is also Gaussian. We can find $f_w(w)$ by finding the mean and variance of $\mathbf{w}$.

$$\begin{aligned} E[\mathbf{w}] &= E[\mathbf{x} - \mathbf{y}] \\ &= E[\mathbf{x}] - E[\mathbf{y}] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \sigma_{\mathbf{z}}^2 &= E[\mathbf{z}^2] \\ &= E[(\mathbf{x} - \mathbf{y})^2] \\ &= E[\mathbf{x}^2] + E[\mathbf{y}^2] - 2E[\mathbf{x}]E[\mathbf{y}] \\ &= 2\sigma^2 \end{aligned}$$

So,

$$f_w(w) = \frac{1}{\sqrt{2\pi 2\sigma^2}} e^{-\frac{w^2}{4\sigma^2}}$$

Thus,

$$\begin{aligned} E[\mathbf{z}] &= \int_{-\infty}^{\infty} |w| f_w(w) dw \\ &= 2\int_{0}^{\infty} w \frac{1}{\sqrt{2\pi 2\sigma^2}} e^{-\frac{w^2}{4\sigma^2}} dw \\ &= \frac{2\sigma}{\sqrt{\pi}} \end{aligned}$$

$E[\mathbf{z}^2]$ is already obtained, which is

$$E[\mathbf{z}^2] = 2\sigma^2$$

5. Use the moment generating function, show that the linear transformation of a Gaussian random vector is also Gaussian.

Proof:
Let $\mathbf{x}$ be a $n \times 1$ real random Gaussian vector, then the density function is

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\mathbf{C_x}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m_x})^T \mathbf{C_x}^{-1}(\mathbf{x}-\mathbf{m_x})}$$

Let $\mathbf{s}$ be a $n \times 1$ real vector, then the moment generating function of $\mathbf{x}$ is

$$\begin{aligned} G_x(\mathbf{s}) &= E[e^{\mathbf{s}^T\mathbf{x}}] \\ &= \int e^{\mathbf{s}^T\mathbf{x}} \frac{1}{(2\pi)^{n/2}|\mathbf{C_x}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m_x})^T \mathbf{C_x}^{-1}(\mathbf{x}-\mathbf{m_x})} d\mathbf{X} \\ &= e^{\mathbf{m_x}^T\mathbf{s}+\frac{1}{2}\mathbf{s}^T\mathbf{C_x}\mathbf{s}} \end{aligned}$$

Let $\mathbf{A}$ be a linear transform of $\mathbf{x}$

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Then

$$C_y = \mathbf{A}\mathbf{C_x}\mathbf{A}^T$$

The moment generating function of $\mathbf{y}$ is

$$\begin{aligned} G_y(\mathbf{s}) &= E[e^{\mathbf{s}^T\mathbf{y}}] \\ &= E[e^{\mathbf{s}^T\mathbf{A}\mathbf{x}}] \\ &= E[e^{(\mathbf{A}^T\mathbf{s})^T\mathbf{x}}] \end{aligned}$$

Using the moment generating function of $\mathbf{x}$, we have

$$\begin{aligned} G_y(\mathbf{s}) &= e^{\mathbf{m_x}^T(\mathbf{A}^T\mathbf{s})+\frac{1}{2}(\mathbf{A}^T\mathbf{s})^T\mathbf{C_x}(\mathbf{A}^T\mathbf{s})} \\ &= e^{\mathbf{m_y}^T\mathbf{s}+\frac{1}{2}\mathbf{s}^T\mathbf{C_y}\mathbf{s}} \end{aligned}$$

which has the same form of $G_x(\mathbf{s})$.
So, $\mathbf{y}$ is also Gaussian.

6. Let $\{x_k(n)\}_{k=1}^4$ be four IID random variables with exponential distribution with α = 1.

$$y_k(n) = \sum_{l=1}^{k} x_l(n), 1 \le k \le 4$$

(a) Determine and plot the pdf of $y_2(n)$
(b) Determine and plot the pdf of $y_3(n)$
(c) Determine and plot the pdf of $y_4(n)$
(d) Compare the pdf of $y_4(n)$ with that of the Gaussian density.

Answer: Let

$$f_x(x) = e^{-x}U(x)$$

The characteristic function of $f_x(x)$ is

$$\phi_x(\omega) = \frac{1}{1 - j\omega}$$

Since $x_1(n), \cdots, x_k(n)$ are i.i.d.,

$$f_{y_k(n)}(y) = f_x(y) * \cdots * f_x(y)$$

Evaluating both sides by the characteristic functions, we have

$$\phi_{y_k(n)}(\omega) = E[e^{j\omega y_k(n)}] = \prod_{l=1}^{k} \phi_{x_l(n)}$$

So,

$$\phi_{y_k(n)}(\omega) = \left(\frac{1}{1 - j\omega}\right)^k$$

whose inverse Fourier transform yields the pdf of $y_k(n)$

$$f_{y_k(n)}(y) = \frac{y^{k-1}e^{-y}}{(k-1)!}U(y)$$

This expression holds for any positive integer k, including $k = 2, 3, 4$.

7. The mean and covariance of a Gaussian random vector $\mathbf{x}$ are given by, respectively,

$$\mu_x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\mathbf{\Gamma}_x = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Plot the 1σ, 2σ, and 3σ concentration ellipses representing the contours of the density function in the $(x1, x2)$ plane. $Hints$: The radius of an ellipse with major axis a (along $x1$) and minor axis $b < a$ (along $x2$) is given by

$$r^2 = \frac{a^2b^2}{a^2sin^2\theta + b^2cos^2\theta}$$

where $0 \leq \theta \leq 2\pi$. Compute the $1\ \sigma$ ellipse specified by $a = \sqrt{\lambda_1}$ and $b = \sqrt{\lambda_2}$ and then rotate and translate each point $\mathbf{x}^(i) = [x_1^{(i)}x_2^{(i)}]$ using the transformation $\mathbf{w}^{(\mathbf{i})} = \mathbf{Q_x x^i} + \mu_\mathbf{x}$.

Answer:

$$\Gamma_x^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

So,

$$\begin{aligned} f_{\mathbf{x}}(\mathbf{x}) &= \frac{1}{2\pi|\Gamma_x|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu_x)^T \Gamma_x^{-1}(\mathbf{x}-\mu_x)} \\ &= \frac{\sqrt{3}}{4\pi} e^{-\frac{2}{3}[(x_1-1)^2-(x_1-1)(x_2-2)+(x_2-2)^2]} \end{aligned}$$

Let

$$g(x_1, x_2) = (x_1 - 1)^2 - (x_1 - 1)(x_2 - 2) + (x_2 - 2)^2$$

The linear transform

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

is a rotation of 45^o of the original axes.

$$g(x_1, x_2) = \frac{w_1^2}{2} + \frac{w_2^2}{\frac{2}{3}}$$

So,

$$a^2 = 2 \qquad b^2 = \frac{2}{3}$$

So, the radius of the ellipse is

$$r^2 = \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta} = 1$$

$$r = 1$$

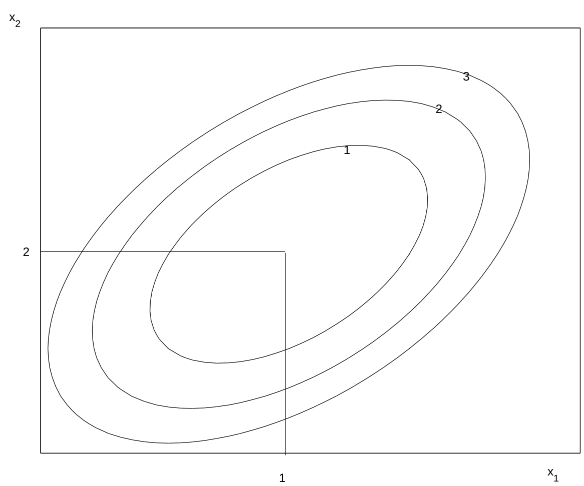
The concentration ellipse of $k\sigma$ (kr) is thus

$$\frac{w_1^2}{2} + \frac{w_2^2}{\frac{2}{3}} = k$$

or

$$(x_1 - 1)^2 - (x_1 - 1)(x_2 - 2) + (x_2 - 2)^2 = k$$

When the function $g(x_1, x_2)$ is chosen differently, the figure will be different. But the orientation of the ellipses are the same.


x2
3
2
1
2
1
x1