1. Let $f_x(t)$ be symmetric about 0. Prove that μ is the expected value of a sample distributed according to $f_{x-\mu}(t)$.

2. The complimentary cumulative distribution function is defined as $Q_x(x) = 1 - F_x(x)$, or more explicitly in the zero mean, unit variance Gaussian distribution case as

$$Q_x(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}t^2\right) dt.$$

Show that

$$Q_x(x) \approx \frac{1}{\sqrt{2\pi}x} \exp\left(\frac{1}{2}t^2\right) dt.$$

Hint: use integration by parts on $Q_x(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}t} t \exp\left(\frac{1}{2}t^2\right) dt$. Also explain why the approximation improves x as increases.

3. The probability density function for a two dimensional random vector is defined by

$$f_{\mathbf{x}}(\mathbf{x}) = \begin{cases} Ax_1^2 x_2 & x_1, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determine $F_{\mathbf{x}}(\mathbf{x})$ and the value of A.
   (b) Determine the marginal density $f_{x_2}(x)$.
   (c) Are $f_{x_1}(x)$ and $f_{x_2}(x)$ independent? Show why or why not.

4. Consider the two independent marginal distributions

$$f_{x_1}(x) = \begin{cases} 1 & 0 \leq x_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{x_2}(x) = \begin{cases} 2x & 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event $x_1 \leq x_2$.

   (a) Find and sketch $f_{\mathbf{x}}(\mathbf{x})$.
   (b) Determine $Pr\{A\}$.
   (c) Determine $f_{\mathbf{x}|A}(\mathbf{x}|A)$. Are the components independent, i.e., are $f_{x_1|A}(x|A)$ and $f_{x_2|A}(x|A)$ independent?

5. The entropy $\mathcal{H}$ for a random vector is defined as $-E\{\ln f_{\mathbf{x}}(\mathbf{x})\}$. Show that for the complex Gaussian case

$$\mathcal{H} = N(1 + \ln \pi) + \ln |\mathbf{C}_x|.$$

Determine the corresponding expression when the vector is real.

6. Let

$$\begin{aligned} x &= 3u - 4v \\ y &= 2u + v \end{aligned}$$

where u and v are unit mean, unit variance, uncorrelated Gaussian random variables.

(a) Determine the means and variances of x and y.

(b) Determine the joint density of x and y.

(c) Determine the conditional density of y given x.

7. Consider the orthogonal transformation of the correlated zero mean random variables x_1 and x_2

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note $E\{x_1^2\} = \sigma_1^2$, $E\{x_2^2\} = \sigma_2^2$, and $E\{x_1 x_2\} = \rho\sigma_1\sigma_2$. Determine the angle θ such that y_1 and y_2 are uncorrelated.

8. The covariance matrix and mean vector for a real Gaussian density are

$$\mathbf{C}_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

and

$$\mathbf{m}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Determine the eigenvalues and eigenvectors.

(b) Generate a mesh plot of the distribution using MATLAB.

(c) Change the off-diagonal values to -0.5 and repeat (a) and (b).

9. Let $\{x_k(n)\}_{k=1}^{K}$ be i.i.d. zero mean, unit variance uniformly distributed random variables and set

$$y_K(n) = \sum_{k=1}^{K} x_k(n).$$

(a) Determine and plot the pdf of $y_K(n)$ for $K = 2, 3, 4$.

(b) Compare the pdf's to the Gaussian density.

(c) Perform the comparison experimentally using MATLAB. That is, generate K sequences of $n = 1, 2, \ldots, N$ uniformly distributed samples. Add the sequences and plot the resulting distribution (histogram). Fit the results to a Gaussian distribution for various K and N.