Chapter 1

University of Delaware
Electrical and Computer Engineering Department
ELEG/PHYS667 Magnetism & Spintronics
Prof. Ian Appelbaum

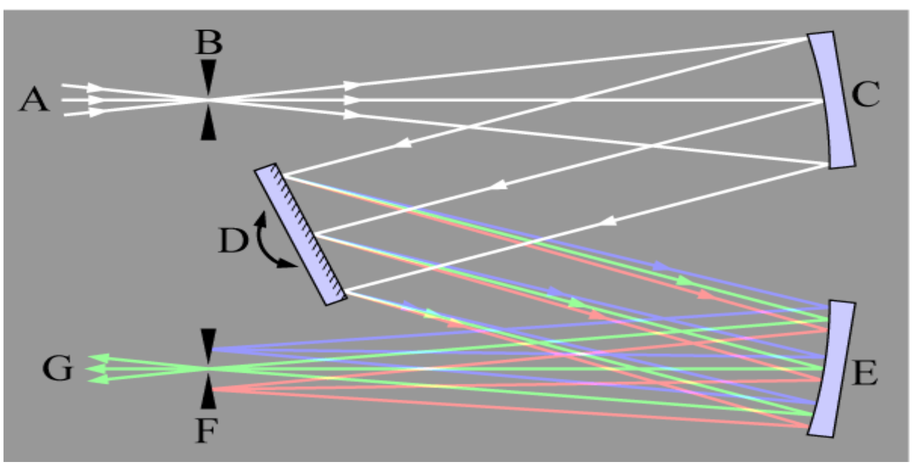
Historically speaking, there were several experiments where electron spin showed itself without being explicitly identified at the time because it predated the birth of quantum mechanics. The most commonly discussed experiments in this context are the Zeeman effect and the Stern-Gerlach experiment.

1 Zeeman effect

The Zeeman effect, first observed in 1896, involves the change in optical spectra from atomic discharge due to the presence of a magnetic field. In this experiment, a glass tube filled with rarefied gas is excited by a high voltage, causing field ionization and electron excitation. The light emitted by atomic vapors in this way consists of sharp spectral lines which can be separated using a spectrometer:

1.1 Spectroscopy

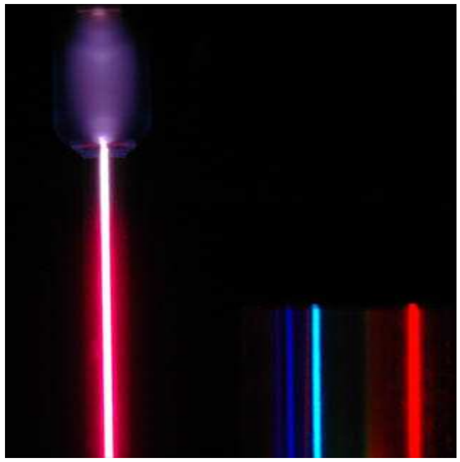
This is a schematic view of the popular Czerny-Turner spectrometer design. Light enters through narrow slits at B and the beam widens out to fill mirror C. This mirror collimates the beam upon reflection and fills the diffraction grating at D. The diffration grating is a special kind of mirror with etched lines close together. Interference from the portions of the beam scattered by each line cause secondary maxima in the reflection pattern of the diffraction grating, at different angles depending on the wavelength of the incident light. These separate beams are focused onto the output slits at F, where a detector can measure the intensity of light at each wavelength by rotating D.



For instance, the light emitted by a hydrogen tube at 500V discharge is shown on the left of the following picture. The spectrum is separated by a spectrometer on the right. We can see sharp spectral lines.

1.2 Why sharp lines?

Although the first successful explanation (by Niels Bohr, for which he won the Nobel prize in 1922) of these sharp lines was not presented until 1913, it is helpful to set the mood for our later discussions.



Bohr was stuck in a classical world, so he used classical physics. He knew from Rutherford (Nobel prize Chemistry 1908) that atoms consisted of a positively charge nucleus in additon to the negatively charged electrons from the work of J.J. Thomson (Nobel prize Physics 1906). So he set up a classical problem with one electron orbiting the oppositely charged nucleus (corresponding to an atom of Hydrogen), balanced by centripetal and electrostatic forces:

$$\frac{mv^2}{r} = \frac{e^2}{r^2} \tag{1}$$

This can be solved for the radius of orbit:

$$r = \frac{e^2}{mv^2} \tag{2}$$

Now, we are after the energies of allowed states in the atom. The total energy is given by

$$E = kinetic + potential = \frac{1}{2}mv^2 + (-\frac{e^2}{r}) \tag{3}$$

Using our expression for the radius r above, we get

$$E = -\frac{1}{2}mv^2. \tag{4}$$

Now, we only need to find the allowed velocities v.

Bohr's insight was that to keep the electron's orbit stable, the angular momentum L must be quantized. Without this constraint, the accelerating electron would radiate energy and collapse into the nucleus.

$$L = mvr = n\hbar \tag{5}$$

where n is an integer greater than zero. We have an expression for r, so we get

$$L = mv\frac{e^2}{mv^2} = \frac{e^2}{v} = n\hbar \tag{6}$$

resulting in the constraint that

$$v = \frac{e^2}{n\hbar} \tag{7}$$

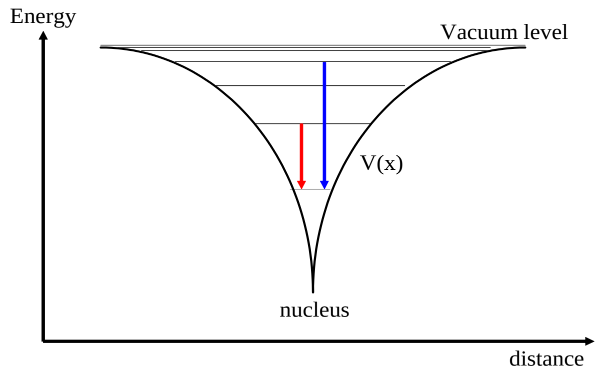
We can substitute this into our expression above for the total energy:

$$E = -\frac{1}{2}m(\frac{e^2}{n\hbar})^2 = -\frac{me^4}{2n^2\hbar^2}. \tag{8}$$

The lowest state has energy

$$-\frac{me^4}{2\hbar^2} = -13.6eV. \tag{9}$$

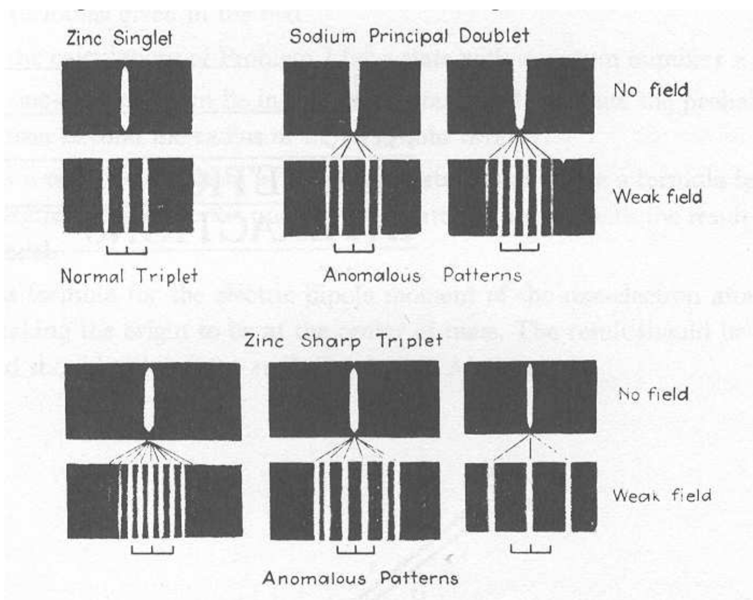
This value is known as a Rydberg, named after a 19th century spectroscopist. States indexed by other values of the *quantum* variable n are shown in the next figure.



We can use this model to explain the features seen in the spectroscopy of atomic emission: Light emission is a process by which electrons in the atom make transitions from a higher energy state to a lower one. The energy (proportional to frequency) of the light is dependent on the energetic difference between the initial and final states.

1.3 Observation

Pieter Zeeman was interested in the Kerr effect, the rotation of polarized light upon reflection from a magnetized surface. He set up his light source near a magnet and using a spectrometer, found that the otherwise spectrally sharp lines appeared blurred. Upon closer inspection with higher resolution and higher magnetic fields, he found that the lines had actually *split*!



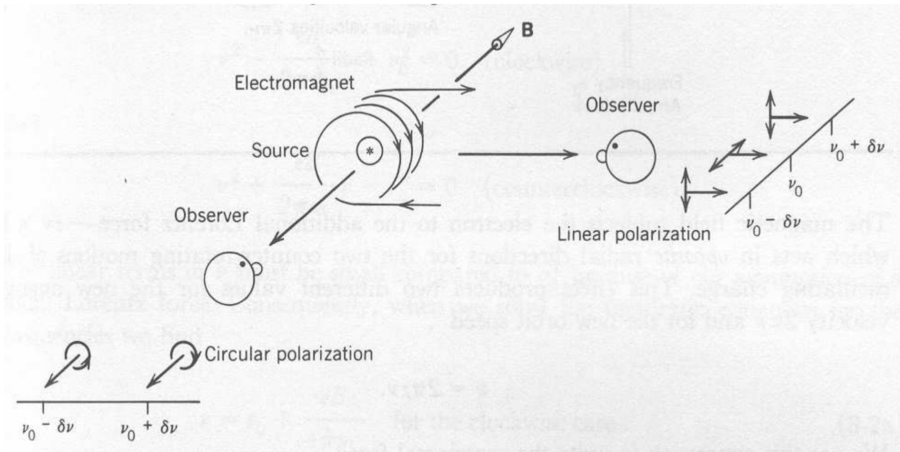
Sometimes the lines split into a triplet in a magnetic field, but more often they split into more. If the lines were observed along the direction of magnetic field, the center (unshifted) line disappeared!

Zeeman's countryman (they were Dutch), Henrick Lorentz, was the first to propose a theory for this splitting. He knew that light was radiated by accelerating charge, so he suggested that the source of the atomic emission was a charge e with mass m oscillating on a spring, with spring constant k. Linear oscillations in the plane perpendicular to the magnetic field B could be decomposed into coherent superpositions of circular orbits traveling clockwise and counter-clockwise. Due to the Lorentz force,

$$F = ev \times B, \tag{10}$$

the magnetic field has an opposite effect on each oppositely-traveling orbit. He balanced the forces: spring force plus Lorentz force equals the centripetal force:

$$kr \pm evB = \frac{mv^2}{r} \tag{11}$$



where the + is for clockwise orbiting charge and − is for counter-clockwise. The charge moves a linear distance equal to the circumference of the orbit with frequency ν per second, so the velocity v is

$$v = 2\pi r\nu. \tag{12}$$

This gives

$$kr \pm e2\pi r\nu B = \frac{m(2\pi r\nu)^2}{r} \tag{13}$$

which can be simplified as

$$\frac{k}{4\pi^2 m} \pm \frac{e\nu B}{2\pi m} = \nu^2. \tag{14}$$

Now, note that in the absence of the magnetic field, the mass m would oscillate with natural frequency

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{15}$$

so we have a quadratic equation

$$\nu^2 \mp \frac{eB}{2\pi m}\nu - \nu_0^2 = 0. \tag{16}$$

This can be solved using the quadratic equation:

$$\nu = \frac{\pm\frac{eB}{2\pi m} \pm \sqrt{(\frac{eB}{2\pi m})^2 + 4\nu_0^2}}{2} \tag{17}$$

Now we have to be careful. The limits of validity for our model are constrained by the fact that we assumed a radius unaffected by the magnetic field. This is only approximately true for small field values. Consequently, if B is small, B^2 is negligible. Therefore, we neglect the first term in the square root and conclude

$$\nu = \nu_0 \pm \frac{eB}{4\pi m} \tag{18}$$

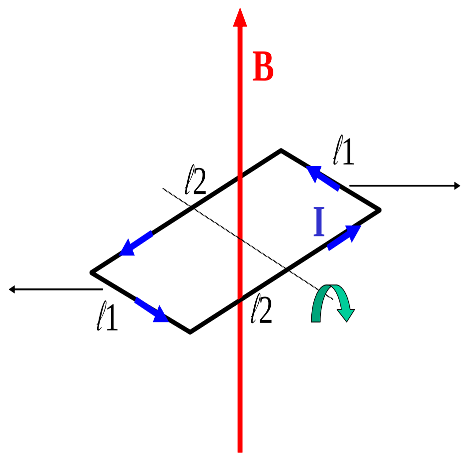
The spectral line at ν_0 due to oscillations in the plane perppendicular to the magnetic field is split into two. Oscillations along B are unaffected (because of the cross product in the Lorentz force) and this shows up at ν_0. Because an accelerating charge does not radiate in the direction parallel to its motion, this unshifted line does not show up if the spectrum is observed in a direction along B. For this theory, Lorentz shared the Nobel prize for physics with Pieter Zeeman in 1902.

Unfortunately, the triplet Lorentz explained (called the normal Zeeman effect) is not the most common splitting. By far the most common splitting is anachronistically called the anomalous Zeeman effect, and Lorentz had no explanation for this.

1.4 The Road to Quantum Mechanics

With over a hundred years of hindsight, we know that the shortcomings of Lorentz's theory are because he didn't have the benefit of quantum mechanics. We will find that a crucial component to the quantum theory's explanation for the Zeeman effect is the concept of magnetic moment, and the energy associated with the magnetic moment in a magnetic field.

Consider the loop of current I in a magnetic field B, shown in the figure.



The Lorentz force on the current elements

$$F = Il \times B \tag{19}$$

is equal and opposite for the two sides of length l_2. However, the Lorentz forces on the other two sides imparts a torque

$$\tau = F \times r \tag{20}$$

on the loop.

Since the current is perpendicular to the field, this is given by

$$\tau = (Il_1B)\frac{l_2}{2}sin(\theta) + (Il_1B)\frac{l_2}{2}sin(\theta) = Il_1l_2Bsin(\theta) \tag{21}$$

The quantity l_1l_2 is the area of the loop, and the product of this area and the current flowing in the loop is called the *magnetic moment*, μ. It is a vector with direction normal to the plane of the loop, in the right-hand sense. We can compactly write the torque on a magnetic moment in a magnetic field as

$$\tau = \mu \times B. \tag{22}$$

Now, the energy is the integral of the torque over an angle:

$$E = \int \tau d\theta = \int \mu B \sin(\theta) d\theta = -\mu B \cos(\theta) \tag{23}$$

which can be compactly written as

$$E = -\mu \cdot B \tag{24}$$

1.5 Orbital moment

The reason we need to learn the above here is that the electron orbiting around the nucleus (in Bohr's model, at least) constitutes a loop of current and so has a magnetic moment.

The area of the orbit is πr^2 and the current is one electron charge $-e$ at ν per second. So the orbital magnetic moment is

$$\mu = \pi r^2 e\nu = -e\left[\pi r^2 \nu\right] \tag{25}$$

In the Bohr model, angular momentum plays a critical role because it is quantized in units of $\hbar$. The velocity of the orbiting electron is $2\pi r\nu$, so we know

$$L = mvr = m(2\pi r\nu)r = 2m\left[\pi r^2 \nu\right]. \tag{26}$$

Notice the terms in brackets... they are the same in both expressions! Therefore, the ratio of magneitc moment to angular momentum, the *gyromagnetic ratio* will not involve this term:

$$\frac{\mu}{L} = \frac{-e}{2m} \tag{27}$$

and since the angular momentum is in units of $\hbar$, it makes sense to express the magnetic moment itself as

$$\mu = \left(\frac{e\hbar}{2m}\right)\frac{L}{\hbar} \tag{28}$$

The term in parenthesis involves only fundamental constants and is known as the *Bohr magneton*, denoted μ_B. It has a value of approximately 5.8×10^{-9} eV/gauss, and is the basic unit of magnetic moment for atomic-scaled systems.

1.6 Zeeman effect revisited

So we can now look at the Zeeman effect from a slightly different viewpoint by considering the direct interaction of the magnetic moment with the magnetic field. If the atomic moment is one Bohr magneton, the maximum energy change to the atom is

$$\Delta E = \pm\mu \cdot B = \pm\frac{e\hbar}{2m}B = h\frac{eB}{4\pi m} = \pm h\nu \tag{29}$$

if

$$\nu = \frac{eB}{4\pi m}. \tag{30}$$

But this is exactly what Lorentz said! This is also particularly evident from the equation of motion for the angular momentum. In linear dynamics we have $F = ma = m\dot{v} = \frac{dp}{dt}$, where p is the momentum. For rotational dynamics we similarly have

$$\frac{dL}{dt} = \tau \tag{31}$$

where L is the angular momentum. Therefore, in this case the equation of motion is

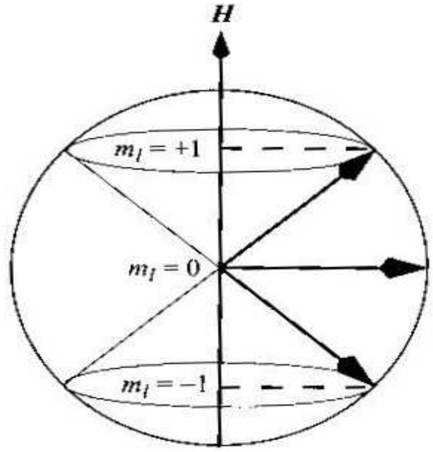
$$\frac{dL}{dt} = \mu \times B = -\frac{eL}{2m} \times B = \frac{eB}{2m} \times L = \omega \times L \tag{32}$$

This is reminiscent of the equation of motion of a mechanically spinning body in a linear force.... Larmor precession at an angular frequency ω. Since $\omega = 2\pi\nu$, we have

$$\nu = \frac{eB}{4\pi m} \tag{33}$$

Now we can insert some results of quantum mechanics. We know that not only is the total angular momentum quantized in discrete values (as Bohr said), but so is the *projection onto any axis*.

The projection quantum numbers range from the angular momentum quantum number in integer steps down to minus the angular quantum number. In the figure above, the angular momentum quantum number l is 1 and the possible projections onto the z-axis are $m_l = 1, 0, -1$. This consequently means that the dot product in $-\mu \cdot B$ takes



on only discrete values leading to discrete energy shifts in the atomic energy levels. The only other conclusion from quantum mechanics we need to insert is that in energetic transitions, one quantum of angular momentum is carried away by the photon. This results in a "selection rule" restricting transitions to those for which the change in the orbital angular momentum quanum number $\Delta l = \pm 1$ and the change in the projection of that angular momentum $\Delta m_l = \pm\hbar$ or 0.

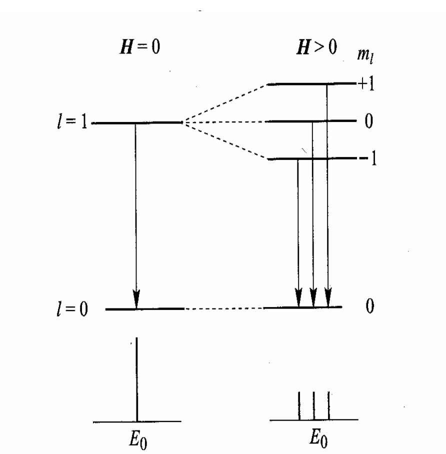
So we can explain the normal Zeeman effect from energy diagram below: The magnetic interaction of the field with the moment causes a splitting. In general, this splits a state with orbital angular momentum quantum number l into $(l,\ l-1,\ \ldots\ 1-l,\ -l) = 2l+1$ states with energy shift

$$\mu_B B m_l \tag{34}$$

when m_l is the projection quantum number. The only allowed photon-emitting transitions are when $\Delta m_l = -1, 0, 1$, so the photon energy can be

$$\Delta E = E_0 + \mu_B B, E_0, E_0 - \mu_B B \tag{35}$$

Which accounts for the observed triplet. However, it does NOT explain the (more common) anomalous Zeeman effect. To explain this, we need to postulate another source of angular momentum intrinsic to the electron. Enter spin...



1.7 Anomalous Zeeman effect

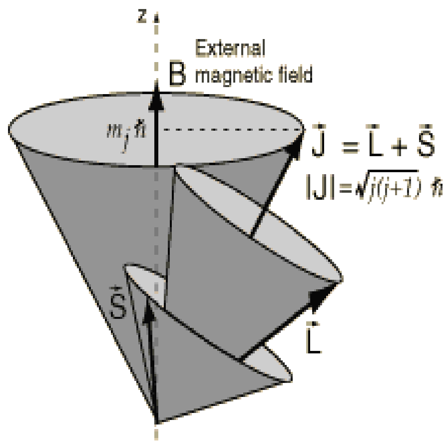
To explain the anomalous Zeeman effect, Goudsmit and Uhlenbeck proposed that there was another source of angular momentum, intrinsic to the electron. The projection of this angular momentum is quantized in half-integer values:

$\pm\hbar/2$, and results in a magnetic moment $\mu_B/2$. Thomas later showed that, for relativistic reasons, when the resulting magnetic moment from this intrinsic "spin" interacted with a magnetic field, the energy is

$$E = 2\frac{\mu_B}{2}B \tag{36}$$

The coefficient of 2 is known as the spin g-factor. In comparison, the orbital g-factor is clearly 1. However, what happens when the atom has both spin and orbital angular momentum? what is g? Can we simply write

$$E = g_L\mu_L B + g_S\mu_S B? \tag{37}$$



Unfortunately, we can't. The problem is that due to the different g-factors for spin (g_S) and orbital (g_L) angular momentum, these vectors precess at different rates about the magnetic field. To make this problem tractable, we define a total angular momentum

$$J = L + S \tag{38}$$

which is a conserved quantity. Now, we project the spin and orbital angular momentum on J. This gives an interaction energy of

$$E = \frac{\mu_B}{\hbar}(g_L L + g_S S)B = \frac{\mu_B}{\hbar}(L + 2S)B = \frac{\mu_B}{\hbar}\left(\frac{(L+2S)\cdot J}{J}\right)\left(\frac{J\cdot B}{J}\right) \tag{39}$$

Since the numerator in the last term is the projection of J on the B axis, and this is quantized in integer intervals from $-J$ to J, we can write

$$J \cdot B = J_z B \tag{40}$$

which simplifies our expression to

$$E = \frac{(L+2S)(L+S)J_z}{J^2}\frac{\mu_B}{\hbar}B = \frac{L^2 + 2S^2 + 3S\cdot L}{J^2}\frac{\mu_z}{\hbar}B \tag{41}$$

Now, we make use of

$$J^2 = (L+S)^2 = L^2 + 2S\cdot L + S^2 \tag{42}$$

to write

$$S \cdot L = J^2 - L^2 - S^2. \tag{43}$$

This gives

$$E = \frac{L^2 + 2S^2 + \frac{3}{2}(J^2 - L^2 - S^2)}{J^2}\frac{\mu_z}{\hbar}B = \frac{3J^2 - L^2 + S^2}{2J^2}\frac{\mu_z}{\hbar}B. \tag{44}$$

1.8 evaluation of J^2, L^2, and S^2

Now, we need to evaluate the squares of the angular momentum operators J^2, L^2, S^2. Since

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{45}$$

we can evaluate this norm by taking the average values of any one projection, for instance:

$$J^2 = 3 < J_z^2 > \tag{46}$$

We evaluate this average by summing the values of the $2j+1$ possibilities for the projection quantum number:

$$< J_z^2 >= \frac{(j\hbar)^2 + ((j-1)\hbar)^2 ...((1-j)\hbar)^2 + (-j\hbar)^2}{2j+1} = \frac{2\sum_0^j x^2}{2j+1}\hbar^2 \tag{47}$$

The sum evaluates to $\frac{j}{6}(j+1)(2j+1)$, so we have

$$< J_z^2 >= \frac{1}{3}j(j+1)\hbar^2 \tag{48}$$

giving

$$J^2 = j(j+1)\hbar^2 \tag{49}$$

1.9 back to the grind

This procedure is general and applies to L and S as well. Therefore, we have

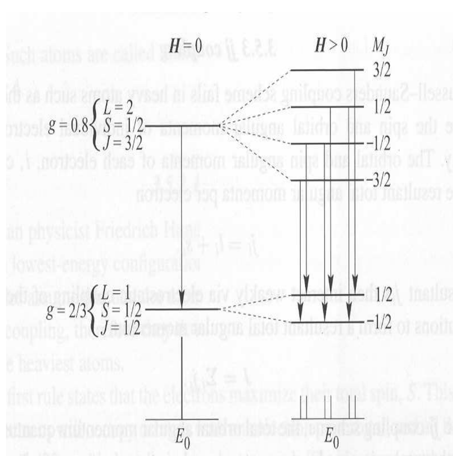
$$E = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} \frac{\mu_z}{\hbar} B. \tag{50}$$

which is usually re-written as

$$E = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right] \mu_B m_J B. \tag{51}$$

Now, notice that the term in brackets functions as an effective g-factor. It was first derived from empirical spectroscopy results by Landé, without the benefit of quantum mechanics. It is therefore called the Landé g-factor. Here, $s = 1/2$, and $j = l+s, l+s-1, ...1-l-s, -(l+s)$, depending on the state.

Eq. 51 is entirely responsible for the anomalous Zeeman effect. It shows that states with different l and j split with different magnitudes in a magnetic field.



Because of this, multiplets higher than triplets (i.e. the normal Zeeman effect) can be seen.

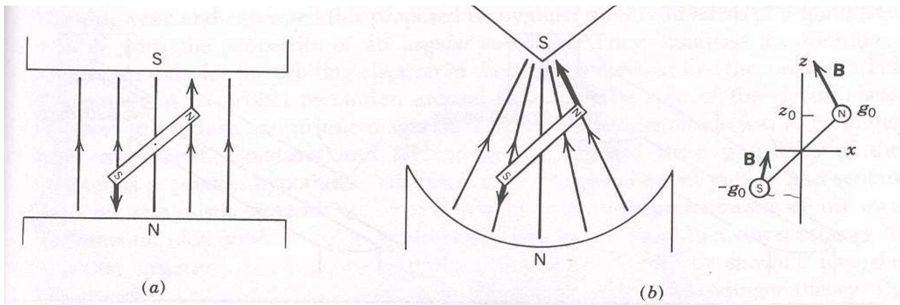
2 Stern-Gerlach Experiment

2.1 Classical magnets in a field gradient

Since the energy of a classical magnetic moment in a magnetic field is

$$E = -\mu \cdot B, \tag{52}$$

we can impart a force on the moment by placing it in a region of spatially varying magnetic field:

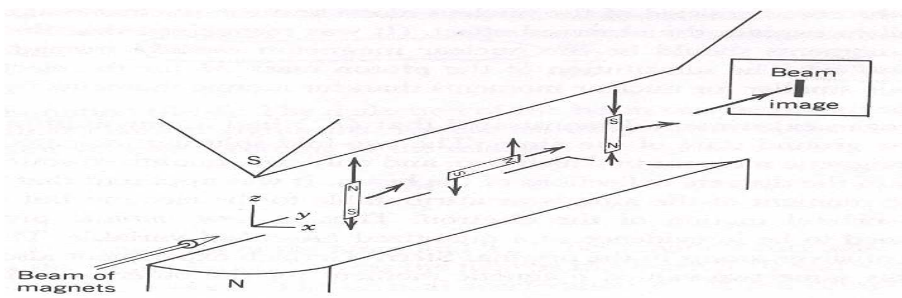


$$F = -\frac{dE}{dz} = +\mu \cdot \frac{dB}{dz}. \tag{53}$$

If the moments are traveling through such a region of length L and with speed v, it will be deflected a distance

$$d = \frac{1}{2}at^2 = \frac{1}{2}\frac{F}{m}\left(\frac{L}{v}\right)^2 = \frac{\mu}{2m}\frac{dB}{dz}\left(\frac{L}{v}\right)^2 \tag{54}$$

along the field gradient.

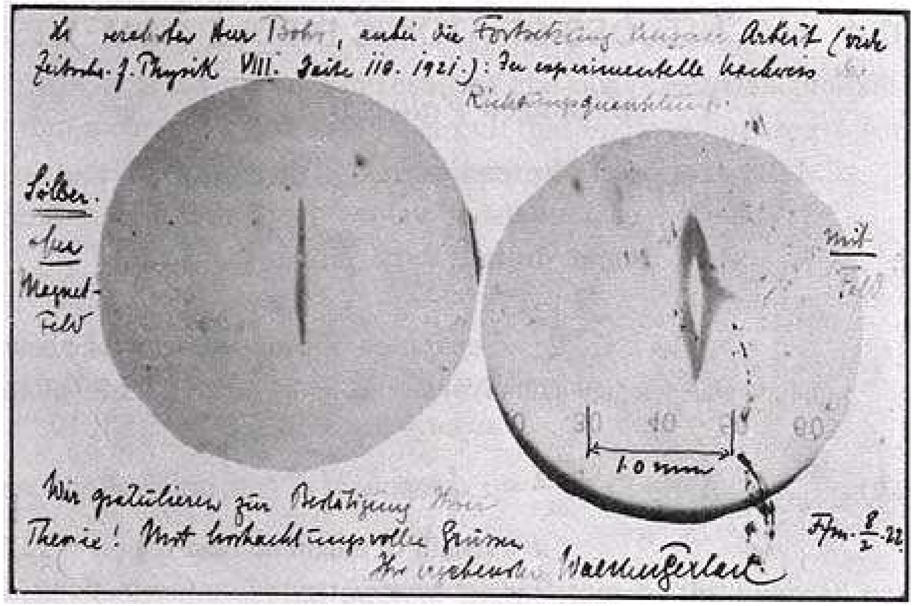


With classical magnets, the orientation of the moments will be randomly distributed. However, as we have seen before, the projection of quantum mechanical moments on any axis is quantized.

2.2 The Experiment

> "Practically all current textbooks describe the Stern-Gerlach splitting as demonstrating electron spin, without pointing out that the intrepid experimenters had no idea it was spin that they had discovered."
> - Bretislav Friedrich and Dudley Herschbach, http://www.physicstoday.org/vol-56/iss-12/p53.html

In 1921, Otto Stern and Walther Gerlach set out to prove Bohr's model by measuring the splitting of a beam of silver (Ag) atoms in a large magnetic field gradient. They knew that there was a magnetic moment associated with the quantized angular momentum Bohr postulated. However, this landmark experiment was amazingly undertaken with false theory and false interpretation: They believed the neutral Ag atoms had angular momentum $l = 1$(wrong), and mistakenly assumed that the allowed quantized projections along B were 1 and -1 ($m = 0$ should have been expected based on their assumption). Fortunately for posterity, their result was re-interpreted in the late 1920s to support Goudsmit's and Uhlenbeck's electron spin hypothesis.



Contrary to Stern and Gerlach's assumption, Ag atoms have a filled 4d (l=2) shell (contributing nothing to the orbital angular momentum because $m = 2$ is cancelled by $m = -2$, $m = 1$ by $m = -1$), and a single unpaired electron in the 5d state: $[\mathrm{Kr}]4\mathrm{d}^{10}5\mathrm{s}^1$. This is a $l = 0$ state, so the atom has NO orbital angular momentum. The unpaired electron DOES carry $S = 1/2\hbar$ intrinsic angular momentum (spin), however. This gives rise to the two split states: $m_s = 1/2$ (which is deflected to one side) and $m_s = -1/2$ (which is deflected to the other).

$$S = \frac{\hbar}{2}, m_S = \pm\frac{1}{2} \tag{55}$$

$$\mu_S = -g_s\mu_B\frac{S}{\hbar} \tag{56}$$

where $g_S = 2$

An initially unpolarized beam of neutral atoms can therefore be *spatially polarized* by passing through a magnetic gradient.