COLORED RANDOM PROJECTIONS FOR COMPRESSED SENSING

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ABSTRACT

This paper describes the reconstruction of sparse signals from a few of incoherent projections based on the theory of Compressed Sensing (CS) when some a priori information about the frequency structure of the signal is available. A new method to design the random projections is introduced that achieves a notably reduced number of measurements while at the same time increases the probability of successful signal reconstruction. The essential improvement is achieved by using colored random projections instead of i.i.d. random projections for the measurements. The proposed approach uses color dithered random matrix as the random measurement matrix since it can achieve desired frequency structure that matches the frequency content of the underlying signal and also has simple spatial form that makes the hardware implementation very convenient. Computationally efficient algorithms to design such color dithered measurement matrix have been developed. Simulation results show the effectiveness of the proposed method.

Index Terms— Compressed Sensing, sparse signal, colored noise, color dithered random projection.

1. INTRODUCTION

Compressed sensing, initially proposed in [1, 2, 3], provides a new way to acquire and represent sparse signals that requires less sampling resource and computational capability than traditional approaches. Given a T sparse signal x ∈ R N on some basis Ψ = [ψ1, ψ2, . . . , ψ N], that is, x can be approximated by a linear combination of T vectors from Ψ, i.e., x ≈ ∑ T i=1 θiψi with T ≪ N, then the theory of Compressed Sensing shows that x can be recovered from M random projections with high probability when M = CT log N ≪ N, where C ≥ 1 is the oversampling factor. The projections are given by y = Φx, where Φ is a M × N random measurement matrix with its rows incoherent with the columns of Ψ. Commonly used random measurement matrices for CS are random Gaussian matrices (Φij ∈ {N(0, 1)/N}), Rademacher matrices (Φij ∈ {±1/√N}) and partial Fourier matrices. In [2], it is shown that a matrix satisfying the incoherent condition is so ubiquitous that “nearly all matrices are CS matrices”.

Signal reconstruction is achieved by solving a l1 norm minimization problem: min ||θ||, subject to ΦΨθ = y, if M ≥ CT log N. V = ΦΨ is called the holographic basis. Minimizing the l1 norm yields solutions that are zero except at a small number of isolated values and can be solved by efficient linear programming algorithms. One method called Basis Pursuit is direct l1 minimization method using, for instance, interior-point method to solve the optimization problem [4]. Matching Pursuit (MP) and Orthogonal Matching Pursuit (OMP) have also been proposed in the literature [3, 5]. These methods are based on greedy algorithms that iteratively reconstruct the sparse signal.

CS relies on the only assumption that the underlying signals are sparse on some basis Ψ and that the measurements are i.i.d. random projections [2]. In practice, many signals not only are sparse, but also have specific structures in basis other than Ψ. For example, a signal sparse in the wavelet domain may have most of its energy concentrated on the low frequency band. Can we design more efficient random projections by exploiting the characteristics of the signal so that the signal can be reconstructed with higher probability using less measurements? This paper addresses this problem in detail.

Our research is motivated by the fact that if we have enough knowledge about the spectrum characteristics of the signal, we can recover the signal with much less measurements [6]. Actually, if the signal x is sparse in the frequency domain and if the locations of these T frequency components are known, then we can recover the signal with only T random projections or samples [6]. On the other hand, it has been shown that passing the signal through a filter with random coefficients has the similar effect of making multiple projections of the signal onto random vectors in CS [7]. If we project the signal to random vectors that have band-pass characteristics (colored random vectors) multiple times, it would be equivalent to get measurements by filtering the signal through a colored random filter and the dominate frequency components of the signal can be enhanced in the measurements. Thus, it is more likely that less projections are needed to capture the salient information of the underlying signal. Colored random projections could help reduce the search space to find the sparse components of the signal in the Basis Pursuit algorithm. Furthermore, an exact knowledge of the signal frequency spectrum is not required for colored random projections.

Random projection vector that has band-pass characteristics and also can be easily implemented via hardware is preferred. This motivates the application of color dithered random vectors for CS. Multi-level color dithered random vectors can also be easily generated by digital halftoning methods [8].

2. COLORED RANDOM PROJECTIONS

2.1. Motivating experiments

Suppose a 1024-point band-pass input signal x(n) as shown in Fig. 1 is to be reconstructed via multiple random projections. Its frequency components are also shown in Fig. 1 where the pass-band of the signal lies between normalized frequency 0.07 and 0.2. Note that
the signal has a sparsity $T = 10$ in the frequency domain. A set of 1024-point random vectors are generated by filtering a normalized white Gaussian noise with a 24-tap band-pass FIR filter $h(n)$ which has a pass-band $0.05 < \omega < 0.2$. The output of the filter is colored random noise that is not evenly distributed in the frequency domain and the noise values are correlated.

2.2. Frequency matching expedites signal recovery

First note that although the energy of the colored random vector is concentrated on its pass-band, it spreads over the whole spectrum. This property can be enhanced if the random vector is dithered with limited levels, as will be discussed shortly. Therefore, it is easy to construct universal colored random measurement matrix so that it is incoherent with any fixed basis $\Psi$.

By matching the frequency components of the colored random vector and the underlying signal, the major frequency components of the signal are enhanced in the measurements and fewer measurements would suffice to contain all the salient information of the signal of interest. Furthermore, the energy of the vectors can be evaluated by integrating over their spectrum. If the peak amplitude of the colored random vectors in the pass-band is the same as the amplitude of the i.i.d. random vectors in the frequency domain, then it is easy to see that colored random projections also lead to less energy consumption in the real applications.

From the signal recovery point of view, colored random projection reduces the search space in the optimization by Basis Pursuit algorithm. More precisely, given the measurement $\Phi x = y$, the optimization process tries to minimize $\|b\|_1$ by searching the space expanded by the rows of measurement matrix $\Phi$. If we study the problem in the frequency domain, then it is clear that the solution only exists in the space expanded approximately by the major frequency components of the colored random vectors and the cost function decreases rapidly in the direction where these major components exist. That is the reason why much less measurements are sufficient to recovery the signal with high probability. Comparing colored random projections with i.i.d. random projections, we know that i.i.d. random projections, although more universal and robust, is not efficient in that it proves a much larger searching space.

3. COLORED RANDOM MEASUREMENT MATRIX

The colored random projection is effective in the CS framework and is simple to generate. However, their spectrum magnitude on the stop-band may be close to 0 leading to singularity problem in the signal reconstruction. This problem can be alleviated if we quantize the colored random vectors. The quantized colored random vectors can be generated by feeding normalized Gaussian noise to a band-pass filter and quantizing the filter output to a finite number of levels. Figure 3(a) shows the frequency representation of a 2-level color dithered random vector which is obtained by feeding normalized Gaussian noise to a band-pass filter with pass-band $0.05 < \omega < 0.2$ and dithering the output to $\pm 1$. It is clear that the desired frequency structure is still preserved after quantization. Furthermore, its robustness is improved. Note that the frequency content for $\omega > 0.2$ is no longer close to zero. Figure 3(b) shows the ensemble average of different spectrum band of different colored dithered projection vectors with 100 realizations for each band.

A more appealing method to generate colored dithered random vectors is to use half-tonning green noise error diffusion structure which appears to be more robust and efficient. It is well known that green noise can generate aperiodic, uncorrelated structure with desired frequency concentration. Figure 4 shows the structure of green noise error diffusion for color dithered random process. In this structure, $h > 0$ is called the hysteresis constant which defines the degree of output dots clustering. Different degree of clustering corresponds to different frequency concentration. The spatial filter $a(n)$, with $\sum_{n=1}^{\infty} a(n) = 1$, is designed in such a way that the output dots are more likely to occur in clusters. $b(n)$ is a normal error
diffusion filter that tries to spread out the errors as homogeneously as possible. The input to the green noise generator is a matrix with uniform density $0 < p < 1$ and each row of the output matrix is a desired color dithered random vector. Note that the hysteresis constant $h$ can be a function of row index $n$ leading to a tunable structure in which the central frequency can vary for each row in the output matrix.

Figure 5(a) shows a 256 x 256 color dithered matrix created by the green noise error diffusion with frequency shifting from high to low along the rows as $h$ increases from 0 to 4. Note that for $h = 0$, high frequency components are present. As $h$ increases, the central frequency shifts to the lower end. The entries are dithered to 1 (white) or -1 (black). Such a matrix is called red-to-blue color dithered matrix. An illustrative example, Fig. 5(b) shows the frequency structure of the 50th row. Note that the energy concentrates on the mid-frequency with a central frequency around 0.2.

When the frequency content of the underlying signal is known, random vector generation reduces to the design of a band-pass filter with the same pass-band of the underlying signal or setting the hysteresis constant $h$ for the desired central frequency.

A more difficult situation arises when the spectrum content of the signal is not known in advance. Similar to the matching filter design, the method we propose here is to use the correlation between the normalized colored random vectors and underlying signal to evaluate the frequency matching between them.

If the green noise error diffusion structure is used, we can vary the hysteresis constant $h$ at coarse scale and generate $L \leq M$ normalized testing vectors $t_i \in R^N$, $i \in \{1, 2, \ldots, L\}$ that covers all the possible frequency range of interest. Next we project the underlying signal $s$ on those testing vectors. The measurement gives a rough description of the frequency content of the signal of interest. The absolute value of these projections are evaluated and the value of $h_{max} = \max_{i \in \{1, 2, \ldots, L\}} |\langle t_i, s \rangle|$ corresponding to the maximal correlation is located. To define the random basis for CS measurements, we can tune the hysteresis constant $h$ around $h_{max}$ to generate more color dithered random vectors with similar frequency structure for measurements. In practice, we can use $N \times N$ red-to-blue color dithered matrix for signal acquisition and recovery. Since the number of measurements $M \leq N$, only a small portion of the rows that have consecutive row indexes need to be used. Although there are central frequency shifting within these rows, the requirements of frequency matching can be approximately satisfied.

The colored random projections generated are suitable for the CS when Basis Pursuit algorithm is used. It is not suitable for MP algorithms if used directly. The reason is that MP prefers uncorrelation between the columns of the holographic basis $V$. The holographic basis from color dithered random projections has more correlation between its columns than that from i.i.d random projections.

Given signal statistics, one approach to apply computationally efficient MP algorithm in colored random projections based CS is to perform unitary transform approximations [9] of the red-to-blue color dithered matrix. Given the autocorrelation matrix $R_{xx}$ of the input signal and the $N$ dimensional red-to-blue color dithered matrix $A$, the desired unitary matrix is given by $B = Q_1Q_2$, where $Q_1$ and $Q_2$ are the unitary matrices derived from the singular value decomposition (SVD) of $C = AR_{xx}C = Q_1AQ_2$. An attractive property of the unitary transform approximation is that it preserves the structure of the original matrix $A$ on some basis [9] which makes frequency matching projections feasible. Given $A$ and $R_{xx}$, $B$ also minimizes the error energy $J = E[|e|^2]e = (A - B)e$. Although $B$ is not a full matrix, it makes the application of MP algorithm in colored random projections possible. Note that the correlation matrix $R_{xx}$ can be estimated adaptively.

4. SIMULATION RESULTS

In this section, several simulations are presented that illustrate the effectiveness of CS based on the color dithered projections. In simulation 1, the signal of interest is a 128-point, band-pass sparse signal $s \in R^N$ with a frequency band $0.21 < \omega < 0.26$. The signal is sparse in the frequency domain with sparsity $T = 10$. The approach to estimate the frequency content of the signal is used to locate the central row index $h_{max}$ of the red-to-blue color dithered matrix to be used for random projections. The reconstruction signal is computed via Basis Pursuit [4]. Figure 6(a) shows the relationship between the number of measurements and the the possibility of successful signal
reconstruction. For comparative purpose, the probability of successful reconstruction as a function of the number of i.i.d. Bernoulli random projections is also shown in Fig. 6(a). At each data point for simulation, 1000 trials are performed and the reconstruction probability is the fraction of the 1000 trials that results in success. A trial is successful when the error $\varepsilon = s - \hat{s}$ between the original signal $s$ and the reconstructed signal $\hat{s}$ has its norm $\|\varepsilon\|_2 \leq 0.01\|s\|_2$. As can be seen from Fig. 6(a), color dithered random projections reduce notably the number of necessary measurements.

In the second simulation, the signal auto-correlation matrix $R_{xx}$ is known and MP algorithm is used for signal reconstruction. The input signal is modelled by the statistic model $s(n) = \sum_{j=0}^{J} \alpha_j \sin(2\pi n (j + 1 / 128))$, where random variable $\alpha_j \sim \text{Uniform} \left( -0.5, 0.5 \right)$ and $\beta_j \sim \text{Bernoulli} \{ 0, 1 \}$. The signal is band-limited and has its main frequency components within $0 < \omega < 0.1$. Unitary matrix transform is performed on an $128 \times 128$ red-to-blue matrix and the resultant unitary matrix is used for color random projections. Figure 6(b) shows the simulation results. It is clear that Matching Pursuit can be as efficient as Basis Pursuit in CS via colored random projections.

The simulation results is shown in Fig. 8. On average, 10 less measurements are achieved by color dithered random projections for the same probability of successful reconstruction compared with i.i.d. random projections. Note that when the signal length $N$ increases, more reduction in the number of measurements is expected.

5. CONCLUSION

It has been shown that when there is a frequency matching between the colored random projection vector and the signal, the sparse signal can be recovered with less measurements compared with i.i.d. random projections. This concept can be extended to the cases where there is strong correlation between the random projection vector and the signal to be measured. In the future, we will extend our current work to 2-D signals and use the similar idea to reconstruct compressible 2-D signal via colored random projections.

6. REFERENCES


